Sociedade Brasileira de Automática (SBA)
XV Simpósio Brasileiro de Automação Inteligente - SBAI 2021, 17 a 20 de outubro de 2021

On the Robustness Properties of Gain-scheduled Unconstrained MPC for LPV Systems ⋆

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Abstract: We assess the robustness qualities of unconstrained gain-scheduled Model Predictive Control (MPC) algorithms for Linear Parameter Varying (LPV) systems. Herein, we generalise finite-horizon robustness analyses of linear time-variant processes, describing the input-output behaviour of the uncertainties through Integral Quadratic Constraints (IQCs). In our case, the uncertainties arise due to the unavailability of the future scheduling variables, from the MPC viewpoint. Accordingly, our analysis procedure computes closed-loop robust induced gains.

Resumo: Avaliamos a robustez de algoritmos irrestritos de Controle Preditivo baseados em Modelo (MPC) aplicados para sistemas Lineares a Parâmetros Variantes (LPV). Generalizamos resultados anteriores de análise de robustez de horizonte finito de processos lineares variantes no tempo, descrevendo o comportamento de entrada-saída das incertezas por meio de Restrições Quadráticas Integrais. Neste caso, as incertezas aparecem devido à indisponibilidade das variáveis de agendamento futuras, do ponto-de-vista do controlador. O procedimento de análise calcula os ganhos induzidos $\mathcal{L}_2$ e $\mathcal{L}_2$-a-Euclidiano para o sistema em malha fechada.

Keywords: Robustness analysis; Model Predictive Control; Linear Parameter Varying Systems; Integral Quadratic Constraints; Dissipativity.

Palavras-chaves: Análise de robutez; Controle Preditivo baseado em modelo; Sistemas Lineares a Parâmetros Variantes; Restrições Quadráticas Integrais; Dissipatividade.

1. INTRODUCTION

In this paper, we are interested in assessing the robustness qualities of Model Predictive Control (MPC) schemes. As argued by Allan et al. (2017), robust MPC is usually synthesised with the use of terminal ingredients and constraints tightening, see e.g. (Santos et al., 2019). Nevertheless, there are no known algorithms for obtaining optimal robust positively invariant terminal set for nonlinear systems, which means that approximations are usually needed, as state Köhler et al. (2020).

Instead of synthesising terminal ingredients, a different approach is brought to focus: we provide a structured robustness analysis tool based on Integral Quadratic Constraint (IQC) arguments in the context of MPC. The toolkit is naturally in compass with the MPC framework, since the IQCs imply in the dissipativity of a finite-horizon quadratic cost function. Basically, if the MPC cost function dissipates w.r.t. the uncertainty description, the IQC arguments hold and thus the closed-loop is ensured robust.

More specifically, our analyses are concerned with "gain-scheduled" MPC algorithms. By this, we mean MPC algorithms conceived for Linear Parameter Varying (LPV) models, but only taking into account the available scheduling information at each sampling instant. A through review on LPV MPC schemes is presented in Morato et al. (2020), which highlights that gain-scheduled algorithms are widely used (e.g. Hanema et al. (2017); Abbas et al. (2018)), especially due to the unavailability of the LPV scheduling parameters along future horizons.

We stress that the typical notions of robustness (e.g. gain/phase margins) are insufficient for LPV systems. We recall the argument from (Seiler et al., 2019): evaluating the stability of a gain-scheduled model can lead to erroneous conclusions, since there exist unstable models $x(k+1) = A(\rho(k))x(k)$ which are stable for frozen values of $\rho(k)$.

The analyses in this work are developed with regard to the structure in Fig. 1: a nominal LPV prediction model G with a state-feedback interconnection κ (the gain-scheduled MPC), and a disturbance interconnection Δ, which represents the prediction mismatches together with possible nonlinearities and uncertainties upon G. As in (Cisneros and Werner, 2018), we describe the predictive controller as a parameter-dependent feedback gain, based on the instaneous values of the scheduling parameter, i.e. $\rho(k)$. Furthermore, as in (Megretski and Rantzer, 1997), we describe the input-output of Δ with IQCs.

⋆ This work has been supported by *CNPq* (304032/2019 − 0) and *ITEA3* European project (15016, *EMPHYSIS*).

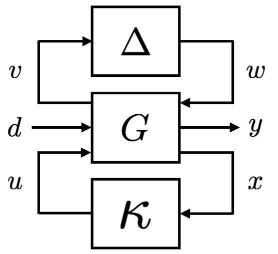

Figure 1. Interconnection $F_u\left(F_l\left(G,\kappa\right),\Delta\right)$ of a nominal prediction model G, a gain-scheduled MPC scheme κ, and uncertainties Δ.

Considering this structure, the main contribution of this paper is the extension and generalisation of the result from (Seiler et al., 2019) to the context of LPV processes regulated under gain-scheduled MPC. We use the time-varying finite-horizon robustness analysis with IQCs in order to provide the robust induced $\mathcal{L}_2$ and $\mathcal{L}_2$-to-Euclidean gains of the closed-loop system $F_u\left(F_l\left(G,\kappa\right),\Delta\right)$.

This paper is organised as follows. In Section 2, we provide the basic gain-scheduled MPC setup, the parameter-dependent state-feedback gain, the nominal performance requirements. In Sec. 3, the main result is presented: we provide bounds on the induced $\mathcal{L}_2$ and $\mathcal{L}_2$-to-Euclidean gains of the closed-loop system. We analyse these bounds through dissipation inequalities and IQCs, which lead to Difference Linear Matrix Inequality (DLMI) remedies. In Sec. 4, we compute the finite-horizon robustness metrics of a benchmark example (a buck-boost DC-DC converter system). General conclusions are drawn in Sec. 5.

Notation: $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n$ denote the set of $n\times m$ real matrices and $n\times n$ real, symmetric matrices, respectively. $[\star]$ denotes a symmetric term that can be inferred from its context. The finite-horizon $\mathcal{L}_2^{N_p}$ norm (of N_p steps) of a signal $v\,:\,[0,N_p]\to\mathbb{R}^n$ is $\|v\|_{2,[0,N_p]}:=\sqrt{\sum_{i=0}^{N_p}\left(v(i)^Tv(i)\right)}$. A bounded norm $\|v\|_{2,[0,N_p]}<+\infty$ implies in $v\in\mathcal{L}_2^{N_p}$. The index set $\mathbb{N}_{[a,b]}$ represents $\{i\in\mathbb{N}\,|\,a\leq i\leq b\}$, with $0\leq a\leq b$. The value of a given variable $v(k)$ at time instant $k+i$, computed based on the information available at instant k, is denoted $v(k+i|k)$.

2. PRELIMINARIES: MPC SETUP AND NOMINAL PERFORMANCE

2.1 Discrete-time LPV Process

Consider the following discrete-time LPV system G:

$$\begin{aligned}x(k+1)&=A(\rho(k))x(k)+B_1(\rho(k))u(k)+B_2(\rho(k))d(k)\,,\\ y(k)&=C(\rho(k))x(k)+D_1(\rho(k))u(k)\,,\end{aligned}\tag{1}$$

where $x\in\mathbb{R}^{n_x}$ represents the system states, $u\in\mathbb{R}^{n_u}$ denotes the control inputs, $d\in\mathbb{R}^{n_d}$ stands for the disturbance inputs, $y\in\mathbb{R}^{n_y}$ are the outputs, and $\rho\in\mathcal{P}\subset\mathbb{R}^{n_p}$ are the scheduling variables. The scheduling set $\mathcal{P}$ is compact, convex and known. Through the sequel, we assume that the model matrices A, B_1, …, and D_1 are bounded affine [1] maps of the scheduling variables ρ. With regard to Fig. 1, Eq. (1) does not yet include the uncertainty variables v and w, which are defined in the sequel.

Note that ρ is only measurable at instant k and unknown for any future instant $k+i$, with $i\geq 1$. Nevertheless, we assume that $\delta\rho(k+1)=\rho(k+1)-\rho(k)$ is bounded to a known compact set $\delta\mathcal{P}$. Note that considering bounded rates of scheduling parameter variations is standard in LPV research and practice (Jungers et al., 2011; Mohammadpour and Scherer, 2012).

2.2 Quadratic Performance Cost

We consider that G is regulated by a state-feedback predictive control scheme κ, as illustrated in Fig. 1. This control scheme is synthesised in order to minimise a quadratic performance cost function J along a prediction horizon of N_p steps, as detailed in (Morato et al., 2020). This parameter-dependent finite-horizon cost function is defined as follows, with $\rho_i=\rho(k+i-1)$:

$$\begin{aligned}J_k&=\overbrace{x(k+N_p)^TP(\rho_{N_p})x(k+N_p)}^{V(x(k+N_p))}\\ &+\underbrace{\sum_{i=1}^{N_p-1}\begin{bmatrix}x(k+i)\\u(k+i-1)\\d(k+i-1)\end{bmatrix}^T\begin{bmatrix}Q(\rho_i)&0&S\\0&R(\rho_i)&0\\S^T&0&T\end{bmatrix}\begin{bmatrix}x(k+i)\\u(k+i-1)\\d(k+i-1)\end{bmatrix}}_{\ell(x(k+i),u(k+i-1),d(k+i-1))}\,.\end{aligned}$$

We name $V(x)$ the terminal offset cost and $\ell(x,u,d)$ the stage cost. $P\succ 0$, $Q\succ 0$, S, R, and T are weighting functions Usually, d is not measurable along the future horizon.

2.3 Nominal Gain-scheduled MPC

MPC works with a receding horizon paradigm: at each instant k, J_k is minimised w.r.t. to a prediction model. Since only $\rho(k)$ is known (and the future values of $\rho(k+i)$ are not), we consider a gain-scheduled approach, based on nominal predictions, in the absence of disturbances and uncertainties. This is, the prediction of the future variables of G are made through Eq. (1) using $\rho(k+i)=\rho(k)$, and $d(k+i-1))=0,\forall i\in\mathbb{N}_{[1,N_p-1]}$. Consider that the MPC is unconstrained, for simplicity. Therefore, it generates a parameter-dependent state-feedback control $u(k)=\kappa(\rho(k))x(k)$, as demonstrated in (Jungers et al., 2011, Theorem 4.4) and (Cisneros and Werner, 2020, Theorem 2). Through the sequel, we assume that $\kappa(\rho(k))$ is known. In practice, this is a stabilising feedback gain $\kappa(\rho)$ found to ensure that the closed-loop dynamics $(A(\rho)+B_1(\rho)\kappa(\rho_k))$ are nominally stable for all $\rho,\rho_k\in\mathcal{P}$.

2.4 Closed-Loop System

The closed-loop system $G_\pi:=F_l(G,\kappa)$ derived from the lower interconnection the LPV system G and the gain-scheduled MPC κ in Fig. 1 implies in the following nominal dynamics:

$$\begin{aligned}x(k+1)&=A_\pi(\rho_k)x(k)+B_2(\rho_k)d(k)\,,\\ y(k)&=C_\pi(\rho_k)x(k)\,,\end{aligned}\tag{2}$$

[1] Other scheduling dependencies could be considered. We choose an affine representation for notation simplicity only.

ISSN: 2175-8905 DOI: 10.20906/sbai.v1i1.2585

where ρ_k denotes the frozen value of the scheduling variable at the sampling instant k. Nominal stability is verified with regard to $A(\rho_k) + B_1(\rho_k)\kappa(\rho_k)$, for all ρ_k, as long as the parameters remain constant along the prediction horizon. For notation compactness, we use $A_\pi(\rho)$ and $C_\pi(\rho)$ in order to indicate, respectively, $A(\rho)+B_1(\rho)\kappa(\rho_k)$ and $C(\rho) + D_1(\rho)\kappa(\rho_k)$, being ρ_k the frozen value of the scheduling variable ρ at instant k.

Nevertheless, as previously discussed, evaluating the robust stability of G_π for fixed values of ρ_k can lead to erroneous conclusions, since, in practice, the scheduling parameters vary along time. Accordingly, we provide robustness assessments due to the variability of $\rho(k + i)$ in Sec. 3. Before that, in any case, present performance metrics and some preliminary results.

2.5 Performance Metrics

As in (Seiler et al., 2019), we consider two specific finite-horizon metrics: the induced $\mathcal{L}_2$ and $\mathcal{L}_2$-to-Euclidean gains of G_π. The first metric is as follows:

$$\|G_\pi\|_{2,[0,N_p]} := \sup \left\{ \frac{\|y\|_{2,[0,N_p]}}{\|d\|_{2,[0,N_p]}} \;\middle|\; \begin{array}{rcl} x(k) &=& 0, \\ d &\neq& 0, \\ d &\in& \mathcal{L}_2^{N_p} \end{array} \right\} .$$

As long as $d \in \mathcal{L}_2^{N_p}$, this gain is finite for any finite prediction horizon N_p; the metric gives the maximal disturbance-to-output energy ratio of the closed-loop system G_π. Note that J_k can be expressed in terms of $\|G_\pi\|_{2,[0,N_p]}$ with a proper choice of the MPC weights (Q, R, S, T). Consider $\rho_i = \rho(k + i - 1)$. Let $\gamma > 0$ be given and take a null terminal cost $V(x)$. Then, with $S = 0$, $T = -\gamma^2\mathbb{I}_{n_d}$, $R = D_1(\rho_i)^T D_1(\rho_i)$, and $Q = C(\rho_i)^T C(\rho_k i) + 2C(\rho_i)^T D_1(\rho_i)\kappa(\rho_k)$, it follows that $J_k = \|y\|^2_{2,[0,N_p]} - \gamma^2\|d\|^2_{2,[0,N_p]}$. Accordingly, $J_k \leq 0, \forall d \in \mathcal{L}_2^{N_p}$ if and only if $\|G_\pi\|_{2,[0,N_p]} \leq \gamma$.

The $\mathcal{L}_2$-to-Euclidean gain of G_π is well-defined since there is no direct disturbance-to-output transfer in Eq. (2). Hence, this metric provides the maximal output y at the last instant of the prediction horizon due to $d \in \mathcal{L}_2^{N_p}$. Therefore, this gain can be used to compute the set of states $x(k+N_p)$ reachable by disturbances of a given norm from $x(k)$. It is defined as follows:

$$\|G_\pi\|_{E,[0,N_p]} := \sup \left\{ \frac{\|y(k + N_p|k)\|_2}{\|d\|_{2,[0,N_p]}} \;\middle|\; \begin{array}{rcl} x(k) &=& 0, \\ d &\neq& 0, \\ d &\in& \mathcal{L}_2^{N_p} \end{array} \right\} .$$

The MPC cost J_k can also be given in terms of $\|G_\pi\|_{E,[0,N_p]}$. Consider $\rho_i = \rho(k + i - 1)$. Let $\gamma > 0$ be given and take $Q = R = S = 0$, $T = -\gamma^2\mathbb{I}_{n_d}$ and $P = C_\pi^T(\rho_{N_p})C_\pi(\rho_{N_p})$. Thus, it follows that $J_k = \|y(k + N_p|k)\|_2^2 - \gamma^2\|d\|^2_{2,[0,N_p]}$ and, thus, $J_k \leq 0, \forall d \in \mathcal{L}_2^{N_p}$ if and only if $\|G_\pi\|_{E,[0,N_p]} \leq \gamma$.

The set of reachable states from the initial conditions $x(k)$ can be given in terms of this induced norm:

$$\mathcal{R}_\beta := \left\{ x(k + N_p) \,:\, \|d\|_{2,[0,N_p]} \leq \beta \right\} . \tag{3}$$

Note that if $C_\pi(\rho)$ is an identity $\mathbb{I}_{n_x}$ and $\|G_\pi\|_{E,[0,N_p]} \leq \gamma$, then $\|x(k + N_p|k)\|_2 \leq \gamma\|d\|_{2,[0,N_p]}$ and $\mathcal{R}_\beta$ is contained within a sphere of radius $\gamma\beta$, see the demonstration in (Seiler et al., 2019).

2.6 A Bounded Real Lemma for $\mathcal{L}_2^{N_p}$ Sequences

Next, we provide a lemma that gives an equivalence between the bounds of the MPC cost J_k and the existence of a solution a Ricatti Differential Inequality (RDI). This result is an extension of the induced $\mathcal{L}_2$ gain of LTV systems in (Başar and Bernhard, 2008) to the finite-horizon LPV case. In Seiler et al. (2019), one can find the equivalence of the RDIs to Ricatti Differential Equations (RDEs), but they will no be addressed in this paper. Furthermore, we stress that there exist corresponding conditions for Linear Time-Invariant (LTI) systems and infinite-horizon performances.

Lemma 1. (Upper bound on J_k and RDI).
Let the MPC weights (P, Q, R, S, T) be given with $T \prec 0$. Then, the following statements are equivalent:

(1) There exists a scalar $\epsilon > 0$ s.t. $J_k \leq -\epsilon\|d\|^2_{2,[0,N_p]}, \forall d \in \mathcal{L}_2^{N_p}$.
(2) There exists a scalar $\epsilon > 0$ and a parameter-dependent map $Y : \mathcal{P} \to \mathbb{S}^{n_x}$ such that $Y(\rho(k + N_p - 1)) \succeq P$ and the following RDI holds with $\rho_i = \rho_k + i\delta\rho$ for all $i \in \mathbb{N}_{[1,N_p-1]}$, $\rho_k, \rho_{-1} \in \mathcal{P}$ and $\delta\rho \in \delta\mathcal{P}$:

$$\begin{aligned} &A_\pi^T(\rho_i)Y(\rho_i)A_\pi(\rho_i) - Y(\rho_{i-1}) \\ &+(Q(\rho_{i-1}) + \kappa^T(\rho_k)R(\rho_{i-1})\kappa(\rho_k)) \\ &-(Y(\rho_{i-1})B_2(\rho_{i-1}) + S)T^{-1}(Y(\rho_{i-1})B_2(\rho_{i-1}) + S)^T \\ &\preceq -\epsilon\mathbb{I}_{n_x} . \end{aligned}$$

Proof 1. Apply a Schur complement over $T \prec 0$ and assume there exists a complementary scalar $\breve{\epsilon} > 0$ s.t.:

$$\begin{aligned} &\left[\begin{array}{c|c} A_\pi^T(\rho_i)Y(\rho_i)A_\pi(\rho_i) - Y(\rho_{i-1}) & Y(\rho_{i-1})B_2(\rho_{i-1}) \\ \hline \star & 0 \end{array}\right] \\ &+ \left[\begin{array}{c|c} (Q(\rho_{i-1}) + \kappa^T(\rho_k)R(\rho_{i-1})\kappa(\rho_k)) & S \\ \hline \star & T \end{array}\right] \leq -\breve{\epsilon}\mathbb{I}_{n_x} . \end{aligned} \tag{4}$$

Let $x(k)$ be a solution of the closed-loop LPV system G_π, departing form the initial condition $x(0) = 0$ and moving due to the load disturbance $d \in \mathcal{L}_2^{N_p}$. Consider the storage function $V(x) = x^T Y(\rho)x$ and use $\breve{Q}(\rho_{i-1}) = (Q(\rho_{i-1}) + \kappa(\rho_k)^T R(\rho_{i-1})\kappa(\rho_k))$. Left- and right-multiply Ineq. (4) by $[x(k+i)\; d(k+i)]^T$ and its transpose, respectively, to obtain:

$$\begin{aligned} &V(x(k + 1)) - V(x(k)) + \breve{\epsilon} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix}^T \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \\ &+ \begin{bmatrix} x(k + i) \\ d(k + i) \end{bmatrix}^T \begin{bmatrix} \breve{Q}(\rho_{i-1}) & S \\ S^T & T \end{bmatrix} \begin{bmatrix} x(k + i) \\ d(k + i) \end{bmatrix} \leq 0 . \end{aligned} \tag{5}$$

Applying a sum over Ineq. (5) from sampling instants k to $k + N_p - 1$ yields:

$$V(x(k+N_p)) - V(x(0)) + \check{\epsilon} \left\| \begin{bmatrix} x \\ d \end{bmatrix} \right\|^2_{2,[0,N_p]}$$
$$+ \sum_{i=1}^{N_p-1} \begin{bmatrix} x(k+i) \\ d(k+i-1) \end{bmatrix}^T \begin{bmatrix} \check{Q}(\rho_{i-1}) & S \\ S^T & T \end{bmatrix} \begin{bmatrix} x(k+i) \\ d(k+i-1) \end{bmatrix} \leq 0\,.$$

Applying the boundary conditions $Y(\rho(k+N_p-1)) \succeq P$ and $x(k) = 0$ implies in $J_k \leq -\check{\epsilon}\|d\|^2_{2,[0,N_p]}$, which is a valid bound for any $d \in \mathcal{L}_2^{N_p}$. This concludes the proof.

Nominal performance of the gain-scheduled MPC is ensured if there exists a parameter-dependent $Y(\rho)$ map that satisfies the RDI in Lemma 1. Nevertheless, the satisfaction of this inequality must hold for all scheduling parameter variations. A simple option is to verify the RDI over a sufficiently dense grid over $\delta\rho \in \delta\mathcal{P}$ for each ρ_k and, then, re-check it over a denser grid in this three dimensional plane, see e.g. (Cisneros and Werner, 2020).

3. MAIN RESULT: ROBUSTNESS AND IQCS

As previously discussed, we consider a gain-scheduled MPC synthesis, for which the feedback gain is parameter-dependent on the instantaneous scheduling value, i.e. $u(k) = \kappa(\rho_k)x(k)$. Nevertheless, the scheduling parameters vary along the prediction horizon, which means that the nominal closed-loop model in Eq. (2) is only valid for $\rho(k+i) = \rho_k, \forall i \in \mathbb{N}_{[1,N_p-1]}$.

We consider the following uncertain model, which describe the real system trajectories:

$$\begin{aligned} x(k+1) &= A_\pi(\rho_k)x(k) + B_2(\rho(k))d(k) + w(k)\,, \\ y(k) &= C_\pi(\rho(k))x(k)\,, \end{aligned} \tag{6}$$

where $v(k) = x(k)$, $x_w(0) = 0$, and:

$$\begin{aligned} x_w(k+1) &= A_\pi(\rho(k))x_w(k) + A_\pi(\rho(k)-\rho_k)v(k)\,, \\ w(k) &= C_w x_w(k)\,. \end{aligned} \tag{7}$$

Through the sequel, we refer to Δ as uncertainty, since it may encompass not only the model-process mismatches along the prediction horizon N_p, but other kinds of perturbations as well, such as memory-less nonlinearities, delays, etc, see (Scherer, 2001). In compact notation, we use $G_\Delta := F_u(G_\pi, \Delta)$. As in previous papers, we consider that G_Δ is well-posed, refer to (Megretski and Rantzer, 1997, Definition 1). We note that G_Δ can be given in an LPV state-space realisation as follows:

$$\begin{aligned} x_\Delta(k+1) &= A_\Delta(\rho(k))x_\Delta(k) + B_{\Delta,1}(\rho(k))w(k) \\ &\quad + B_{\Delta,2}(\rho(k))d(k)\,, \\ v(k) &= C_{\Delta,1}x_\Delta(k) + D_{\Delta,1,1}(\rho(k))w(k) \\ &\quad + D_{\Delta,1,2}(\rho(k))d(k)\,, \\ y(k) &= C_{\Delta,2}x_\Delta(k) + D_{\Delta,2,1}(\rho(k))w(k) \\ &\quad + D_{\Delta,2,2}(\rho(k))d(k)\,, \end{aligned} \tag{8}$$

where $x_\Delta := [x\,,\, x_w]^T$ and $w := \Delta v$. In our analyses, we consider that $\|\Delta\|_\infty \leq 1$. This bound can be satisfied by an adequate choice of the matrices in Eq. (8). Note that w is an uncertainty input to the system.

3.1 *Integral Quadratic Constraints*

In order to demonstrate robust stability of the interconnection represented by G_Δ, we will use the IQC framework from (Megretski and Rantzer, 1997), adapted to the discrete-time LPV context. We use these constraints to describe the input-output behaviour of the uncertainty interconnection Δ. As in Seiler et al. (2019), we use a time-domain representation of such IQCs, but we stress that frequency-domain constraints could also have been used, see e.g. Seiler (2014).

The IQC formulation is set upon the outputs z of an LTI filter Ψ, which has null initial conditions and is fed by the uncertainty-related variables v and w. We enforce an IQC relationship over z, considering the dynamics along the whole the prediction horizon N_p. The filter dynamics are as follows:

$$\begin{aligned} x_\psi(k+1) &= A_\psi x_\psi(k) + B_{\psi,1}v(k) + B_{\psi,2}w(k)\,, \\ z(k) &= C_\psi x_\psi + D_{\psi,1}v(k) + D_{\psi,2}w(k)\,. \end{aligned} \tag{9}$$

The IQC is defined as follows: Let $\psi \in \mathbb{RH}_\infty^{n_z\times(n_v+n_w)}$ and $M : \mathcal{P} \to \mathbb{S}^{n_z}$. An operator Δ satisfies the IQC denoted $\mathcal{I}(\Psi, M)$ if the following inequality holds for all $v \in \mathcal{L}_2^{N_p}$ with the interconnection $w := \Delta v$:

$$\sum_{i=0}^{N_p-1} z(k+i)^T M(\rho(k+i))z(k+i) \geq 0\,. \tag{10}$$

Specifically, we say that $\Delta \in \mathcal{I}(\Psi, M)$ if Δ satisfies Ineq. (10). In the following subsection, we show the corresponding IQC formulation for the case when Δ represents only the LPV model-process mismatches.

3.2 *IQC for Prediction Uncertainties*

Take Δ with $\|\Delta\|_\infty \leq 1$ as gives Eq. (7) with $C_w = \mathbb{I}_{n_v}$ and and $n_v = n_x$. Since $\rho(k+i) = \rho_k + i\delta\rho(k)$ is a time-varying real parameter, we use $z := x_\psi$, with $x_\psi = x_w(0)$. Note that, for $k = 0$, we have no prediction errors since ρ_k and ρ_{-1} are known. Accordingly, we obtain:

$$\begin{aligned} x_w(k+1) &= \left(A(\rho(k) + B_1(\rho(k)\kappa(\rho_k)\right) x_w(k) \\ &\quad + \left(A(\rho(k)-\rho_k) + B(\rho(k)-\rho_k)\kappa(\rho_k)\right) x(k)\,. \end{aligned} \tag{11}$$

In this case, since $A_\pi(\rho)$ is nominally stable for all $\rho \in \mathcal{P}$, we can choose Δ as an LPV filter. Let us define $\Psi := \mathrm{diag}\{\mathbb{I}_{n_x}, \mathbb{I}_{n_x}\}$ and $M(\rho(k+i)) := \mathrm{diag}\{m_{11}(\rho(k+i))\,, -m_{11}(\rho(k+i))\}$ with $m_{11} : \mathcal{P} \to \mathbb{R}$ such that $m_{11}(\rho(k+i)) := |\rho(k+i)| \geq 0, \forall i \in \mathbb{N}_{[0,N_p]}$. Then, $\Delta \in \mathcal{I}(\Psi, M)$. For further details, refer to (Megretski and Rantzer, 1997, Section VI.A) and (Seiler et al., 2019, Example 5).

We stress that the size of the prediction horizon N_p plays a key role in the uncertainty description, since $(\rho(k) - \rho_k)$ grows with N_p. Note that $\rho(k+i) - \rho_k = \sum_{j=1}^{i} \delta\rho(k+j) \in i\delta\mathcal{P}$. This means that as the horizon size N_p increases, we should expect the uncertainty's effect to become more significant. This implies in less robustness (larger induced $\mathcal{L}_2$ and $\mathcal{L}_2$-to-Euclidean gains for G_Δ, for instance). Also, it is reasonable to expect that as N_p grows, the robustness metrics should become closer to the worst-case infinite-horizon result. In theory, we expect to obtain $\lim_{N_p \to +\infty} \|G_\pi\|_{2,[0,N_p]} \to \|G_\pi\|_2$.

Complementary, we point out that a thorough library of IQCs is provided in (Megretski and Rantzer, 1997, Section VI), for the most diverse kinds of uncertainty descriptions. The discussions on different IQC characterisations presented in (Fetzer et al., 2017; Scherer and Veenman, 2018) are also very welcome to elucidate the context.

3.3 Robust Induced Gains

We will proceed by analysing the robustness of G_Δ using the auxiliary filter Ψ. Thus, in the sequel, the uncertainty description $w := \Delta v$ plays no role, but the focus is cast over the dynamics z.

Firstly, we convert the dynamics of the uncertain model G_Δ (Eq. (6)) and the LTI filter Ψ (Eq. (9)), which yields:

$$\begin{aligned} x_e(k+1) &= A_e(\rho(k))x_e + B_e(\rho(k))\mu(k)\,, \\ z(k) &= C_{e,1}x_e(k) + D_e\mu(k)\,, \\ y(k) &= C_{e,2}x_e(k)\,, \end{aligned} \tag{12}$$

where $x_e = [x^T\,,\, x_\psi^T]^T$, $\mu = [w^T\,,\, d^T]^T$ and matrices are:

$$A_e(\rho) = \left[\begin{array}{c|c} A(\rho) + B(\rho)\kappa(\rho_k) & 0 \\ \hline B_{\psi,1} & A_\psi \end{array}\right],$$

$$B_e(\rho) = \left[\begin{array}{c|c} \mathbb{I} & B_2(\rho) \\ \hline B_{\psi,2} & 0 \end{array}\right], D_e(\rho) = \left[\, D_{\psi,2} | 0 \,\right]^T,$$

$$C_{e,1}(\rho) = \left[\begin{array}{c} D_{\psi,1}^T \\ \hline C_\psi^T \end{array}\right]^T, C_{e,2}(\rho) = \left[\begin{array}{c} (C(\rho) + D_1(\rho)\kappa(\rho_k))^T \\ \hline 0 \end{array}\right]^T.$$

The following Lemmas use the extended model in Eq. (12) and Lemma 1 to provide the robust induced $\mathcal{L}_2$ and $\mathcal{L}_2$-to-Euclidean gains of $F_u(G_\pi, \Delta)$.

Lemma 2. Robust Induced $\mathcal{L}_2$ Gain
Let G be an LPV system defined by Eq. (1), controlled by a gain-scheduled MPC feedback gain $\kappa(\rho_k)$ s.t. the closed-loop dynamics $G_\pi = F_l(G, \kappa)$ are given by Eq. (6). Let $\Delta : \mathcal{L}_2^{N_p} \to \mathcal{L}_2^{N_p}$ be an uncertainty operator. Assume $F_u(G_\pi, \Delta)$ is well-posed and $\Delta \in \mathcal{I}(\Psi, M)$. Consider that there exists scalars $\epsilon, \gamma > 0$ and a parameter-dependent map $Y : \mathcal{P} \to \mathbb{S}^{n_x + n_\psi}$ such that condition $Y(\rho(k + N_p - 1)) \succeq P$ and that the following inequality holds with $\rho_i = \rho_k + i\delta\rho$, for all $i \in \mathbb{N}_{[1,N_p-1]}$, $\rho_k, \rho_{-1} \in \mathcal{P}$ and $\delta\rho \in \delta\mathcal{P}$:

$$\left[\begin{array}{c|c} A_e^T(\rho_i)Y(\rho_i)A_e(\rho_i) - Y(\rho_{i-1}) & Y(\rho_{i-1})B_e(\rho_{i-1}) \\ \hline \star & 0 \end{array}\right] + \left[\begin{array}{c|c} \tilde{Q}(\rho_{i-1}) & S \\ \hline \star & T \end{array}\right] + [\star]^T M \left[\, C_{e_1}(\rho_{i-1}) \;\; D_e(\rho_{i-1}) \,\right] \le -\epsilon\mathbb{I}\,. \tag{13}$$

Then, it follows that $\|F_u(G_\pi, \Delta)\|_{2,[0,N_p]} < \gamma$.

Proof 2. The proof follows directly from (Seiler et al., 2019, Theorem 6), converting the time-varying terms into scheduling parameter dependencies. Furthermore, select $Q(\rho_{i-1}) = C_{e,2}(\rho_{i-1})^T C_{e,2}(\rho_{i-1})$, $S = 0$, $T = -\gamma^2 \text{diag}\{0_{n_w}, \mathbb{I}_{n_d}\}$.

Lemma 3. Robust Induced $\mathcal{L}_2$-to-Euclidean Gain
Let G be an LPV system defined by Eq. (1), controlled by a gain-scheduled MPC feedback gain $\kappa(\rho_k)$ s.t. the closed-loop dynamics $G_\pi = F_l(G, \kappa)$ are given by Eq. (6). Let $\Delta : \mathcal{L}_2^{N_p} \to \mathcal{L}_2^{N_p}$ be an uncertainty operator. Assume $F_u(G_\pi, \Delta)$ is well-posed and $\Delta \in \mathcal{I}(\Psi, M)$. Consider that there exists scalars $\epsilon, \gamma > 0$ and a parameter-dependent map $Y : \mathcal{P} \to \mathbb{S}^{n_x + n_\psi}$ such that condition $Y(\rho_{N_p-1}) \succeq P$ and that Ineq. (13) holds with $\rho_i = \rho_k + i\delta\rho$, for all $i \in \mathbb{N}_{[1,N_p-1]}$, $\rho_k, \rho_{-1} \in \mathcal{P}$ and $\delta\rho \in \delta\mathcal{P}$. Then, it follows that $\|F_u(G_\pi, \Delta)\|_{E,[0,N_p]} < \gamma$.

Proof 3. This Lemma is a mere adaptation of the prior. For such, select $\tilde{Q} = S = 0$, $T = -\gamma^2 \text{diag}\{0_{n_w}, \mathbb{I}_{n_d}\}$ and $P = C_{e,2}(\rho_{N_p-1})^T C_{e,2}(\rho_{N_p-1})$.

Remark 1. Lemmas 2 and 3 provide parameter-dependent RDI solutions that can be used to bound the induced $\mathcal{L}_2$ and $\mathcal{L}_2$-to-Euclidean gains of G_Δ. Other metrics can be used by a proper selection of the matrices in Lemma 1.

4. BENCHMARK EXAMPLE

Consider the DC-DC Buck-Boost converter benchmark model (Lazar et al., 2008):

$$A = \begin{pmatrix} 1 & 0.0541 \\ -0.1033 & 0.9909 \end{pmatrix}, \qquad B_1(\rho) = \begin{pmatrix} 2.619 - \rho_2 \\ 0.24 + \rho_1 \end{pmatrix}, \tag{14}$$

$$\rho = (0.2273x_1\,,\, 0.119x_2)^T\,, \qquad B_2 = I_{n_x}\,, \tag{15}$$

being x_1 the inductor current, x_2 the output tension, u a duty-cycle input signal, and scheduling parameter sets:

$$\mathcal{P} := \left\{\rho \in \mathbb{R}^2 \colon |\rho_1| \le \gamma_1, |\rho_2| \le \gamma_2\right\}.$$
$$\delta\mathcal{P} := \left\{\delta\rho \in \mathbb{R}^2 : |\delta\rho_1| \le 0.086, |\delta\rho_2| \le 0.025\right\}.$$

The system operates subject to additive load disturbance bounded to the box $d \in \mathcal{D} \subset \mathbb{R}^2$ such that $\|d(k)\| \le 0.02, \forall k$. Our results were obtained with Matlab, Yalmip, and SDPT3 in a 2.4 GHz, 8 GB RAM Macintosh computer.

We use the gain-scheduled synthesis from (Cisneros and Werner, 2020, Theorem 2) with unitary tuning weights in order to obtain a parameter-dependent state-feedback MPC gain. Likewise, we use an LQR solution with the same tuning weights in order to obtain the infinite-horizon correspondence. For this system, in closed-loop, the infinite-horizon worst-case induced $\mathcal{L}_2$ gain is of 1.2, while the induced $\mathcal{L}_2$-to-Euclidean gain is of 0.8.

With the aid of Lemmas 2 and 3, we compute the induced robust gains ($\mathcal{L}_2$ and $\mathcal{L}_2$-to-Euclidean) of the closed-loop system with the gain-scheduled MPC feedback $F_u\left(F_l\left(G, \kappa\right), \Delta\right)$, where Δ represents the model uncertainties that arise when using a frozen LTI prediction at each sampling instant. These gains are presented in Fig. 2, which clearly indicates that the uncertainties increase with the size of the horizon. We stress that this is quite logical result, since the real scheduling trajectory $\rho(k+j)$ further differs from the frozen trajectory $\hat{\rho}(k+j) = \rho_k$ for longer predictions (larger N_p).

The induced $\mathcal{L}_2$-to-Euclidean gain also serves to compute the set of reachable states $x(k+N_p)$ with $y = x$ in Eq. (3).

ISSN: 2175-8905

DOI: 10.20906/sbai.v1i1.2585

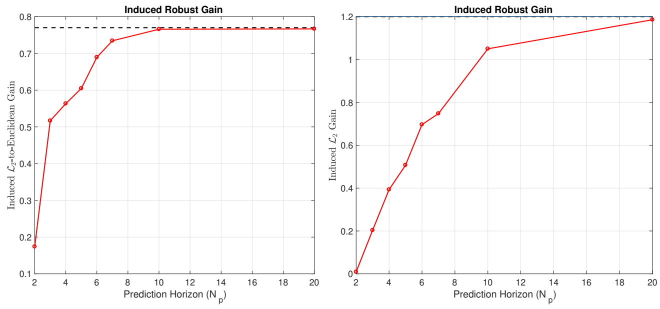

Figure 2. Induced Robust Gains.

Accordingly, consider a control horizon of $N_p = 10$ steps. Then, since $\|G_\pi\|_{E,[0,N_p]} \leq 0.8$ (see Fig. 2) and $\|d\| \leq 0.2$, there exists a terminal set $\mathcal{R}_\beta$ which contains all possible reachable states due to these disturbances.

Considering fifty random disturbance sequences at four different initial conditions $x(0) := [\pm 0.5, \pm 0.5]^T$, Fig. 3 depicts the phase-plane state trajectories converging to the origin. The terminal values $x(N_p)$ are marked with bold circled dots, while the reachable set of states $\mathcal{R}_\beta$ is represented as a blue disk with radius 0.16. Clearly, this set indeed contains all terminal conditions $x(N_p)$, which means that the computed induced robust gain is coherent. In the context of MPC, this robustness metric can be used to compute the region of attraction of the controller.

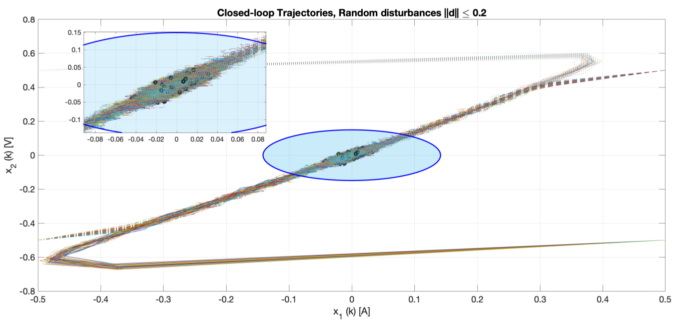

Figure 3. Closed-Loop system trajectories subject to random bounded disturbances.

5. CONCLUSIONS

In this paper, we presented induced $\mathcal{L}_2$ and $\mathcal{L}_2$-to-Euclidean robustness metrics for LPV systems controlled under gain-scheduled unconstrained MPC algorithms. The result is an extension of finite-horizon IQCs for LTV systems. The robust gains are shown to be valid trough a simple example. The same methodology is valid for invariant systems subject to increasing, but limited, uncertainties. For future works, the Lemmas presented herein will be extended for the case of constrained MPCs.

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