Observer and Observer-Based Control Designs with Guaranteed Performance for Linear Discrete-Time Descriptor Systems ⋆

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Abstract:
This paper approaches the problem of Luenberger-like observer design for a class of LTI discrete-time descriptor systems. We present and prove sufficient conditions for the observer design with a guaranteed decay rate where the admissible descriptor systems. In addition, assuming some standard controllability properties, a separation principle is demonstrated considering an observer-based output feedback control law. The observer design is also extended to cope with model disturbances in an H_2 sense. The effectiveness of the proposed methodology is illustrated by numerical examples.

Keywords: Descriptor system, State observer, Linear Matrix Inequalities (LMIs), Separation Principle.

1. INTRODUCTION

The introduction of the state-space approach in late 50's and early 60's made possible to develop powerful mathematical tools for the analysis and control synthesis of linear feedback systems. To deal with more complex dynamic phenomena, the standard state-space representation has been extended to the class of descriptor systems which is also referred in specialized literature as singular systems, implicit systems, generalized state-space representation and differential-algebraic systems. Descriptor models are utilized in many applications such as social-economic and biological systems as well as in many engineering fields (e.g., electrical power systems, aerospace engineering, chemical processes, robotic systems, among others); see, for instance, (Duan, 2010) and references therein.

In modern control theory, the control design is based on a state feedback control law which assumes that the system states are available online to the controller. Unfortunately, in most of practical applications, the task of measuring all system states for feedback purposes is hard or even impossible to accomplish. In such cases, a common strategy is to estimate the system states from the knowledge of a few measurements by means of a state observer (Luenberger, 1966). Hence, the estimation problem of descriptor systems is of great interest. However, most of consolidate techniques for observer design are dedicated to continuous-time systems as, for instance, the seminal work of Kalman and Bucy (1961) and the more recent ones of Koenig (2006), Khalil and Praly (2014), Alma and Darouach (2014), Efimov et al. (2015) and Nguyen et al. (2018), to cite a few.

In the context of discrete-time descriptor systems, several works have addressed the estimation and filtering problems exploiting the fast dynamics observability condition to obtain a standard state-space representation that is more suitable for observer design, such as Wang et al. (2012) which addressed the estimation problem of linear and Lipschitz systems; Boulkroune et al. (2013) and Wang et al. (2015) which dealt with the fault detection and isolation problem; and Wang et al. (2018) and Guo et al. (2019) which proposed interval observers for nonvanishing disturbances. However, none of the latter results have studied the observer-based output feedback problem which is particularly challenging considering that the current estimation provided by these observers is a function of the current measurement.

This work follows the latter observer design approaches but assuming an observer-based output feedback application and considering the free multipliers associated with the fast dynamics observability condition as decision variables. We firstly propose an LMI feasibility problem for designing an observer having a standard state-space representation while guaranteeing a given estimation error

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convergence rate. Then, the observer design is extended to cope with process disturbances considering the H_2 setting, and a separation principle is also established showing that the observer-based output feedback control (i.e., the state-feedback of estimated state variables) can be independently accessed under some mild assumptions. Numerical examples demonstrate that the effectiveness of the observer-based output feedback design as well as the estimation error performance improvement with respect to process disturbances when compared to some existing approaches which consider fixed multipliers associated to the fast dynamics observability condition.

Notation: $\mathbb{C}$ is the set of complex numbers, $\mathbb{R}$ is the set of real numbers, $\mathbb{N}$ and $\mathbb{N}^+$ are respectively the sets of non-negative and positive integer numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of $m \times n$ real matrices, $\|\cdot\|$ is the Euclidean vector norm, I_n is the $n \times n$ identity matrix, 0_n and $0_{m\times n}$ are the $n \times n$ and $m \times n$ matrices of zeros, respectively, and $\mathrm{diag}\{\cdots\}$ denotes a block-diagonal matrix. For a real matrix W, $\mathrm{rank}(W)$ is the rank of W, W^T denotes its transpose and $W^\sharp$ represents its Moore-Penrose pseudoinverse. For a real and square matrix S, $\mathrm{He}(S)$ stands for $S + S^T$ and $S > 0$ ($S \geq 0$) means that S is symmetric and positive-definite (positive semi-definite). For a symmetric block matrix, $\star$ stands for the transpose of the blocks outside the main diagonal block. For a nonnegative integer number k and a vector sequence $f(k)$, its ℓ_2 norm is defined as $\|f(k)\|_2 = \sqrt{\sum_{k=0}^{\infty} f(k)^T f(k)}$.

2. PROBLEM STATEMENT

Consider the following discrete-time linear time-invariant (LTI) descriptor system

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k),\ x(0) = x_0, \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input vector, $y(k) \in \mathbb{R}^p$ is the output vector, and A, B, C and E are known real matrices with appropriate dimensions with E allowed to be singular and satisfying $\mathrm{rank}(E) = r \leq n$.

The nonsingularity of E induces some complexity in the behavior of system (1). For instance, the initial condition x_0 is not an arbitrary vector in $\mathbb{R}^n$, since it has to satisfy the algebraic constraint related to the null space of E (Karampetakis and Gregoriadou, 2014). To better characterize the solvability conditions of (1), the following definitions related to general discrete linear time-varying descriptor systems are introduced.

Definition 1. (Barbosa et al., 2018) Consider the following descriptor system

$$E(k)\xi(k+1) = A(k)\xi(k) + B(k)v(k), \quad \xi(0) = \xi_0, \tag{2}$$

where $\xi \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, and $A(k), B(k)$ are bounded matrices for all $k \in \mathbb{N}$, with ξ_0 being a consistent initial condition. Then:

(a) The system is said to be regular if for any $v(k) \in \mathbb{R}^m$ and ξ_0 there exists a solution $\xi(k)$ for all $k \in \mathbb{N}$ and it is unique.
(b) The system is said to be causal if it is regular and the solution $\xi(k)$ for any ξ_0 and $v(k) \in \mathbb{R}^m$ is a function of ξ_0 and $v(0), \ldots, v(k)$ for all $k \in \mathbb{N}$.
(c) The system is said to be exponentially stable if it is regular and for any ξ_0 and $v(k) \equiv 0$, there exist real scalars $\alpha \geq 1$ and $\beta \in (0, 1)$ such that
$$\|\xi(k)\| \leq \alpha\, \beta^k\, \|\xi_0\|, \ \forall\, k \in \mathbb{N}^+.$$
(d) The system is said to be admissible if it is causal and exponentially stable.

For LTI descriptor systems, the conditions for ensuring the system admissibility are well-known in the literature and are summarized in the following lemma.

Lemma 1. (Zhang et al., 2008) Let x_0 be a consistent initial condition. Then, system (1), or the pair (E, A), is admissible if the following conditions hold:

1) $\det(zE - A)$ is not identically zero;
2) $\deg\{\det(zE - A)\} = r$, with $\deg\{\det(zE - A)\}$ representing the degree of $\det(zE - A)$; and
3) $\rho(E, A) < 1$, with $\rho(E, A)$ standing for the generalized spectral radius.

Before introducing the problem to be addressed in this paper, we assume the following with respect to system (1):

Assumption 1. The initial state x_0 of system (1) is consistent, in the sense that $E_0 A x_0 + E_0 B u(0) = 0$, where $E_0 \in \mathbb{R}^{q\times n}$, with $q = n - r$, is a full row-rank matrix such that $E_0 E = 0$.

Assumption 2. The pair (E, C) satisfies the following:

$$(i)\ \mathrm{rank}\left(\begin{bmatrix} E \\ C \end{bmatrix}\right) = n,$$

$$(ii)\ \mathrm{rank}\left(\begin{bmatrix} (zE - A) \\ C \end{bmatrix}\right) = n,\ \forall\, z \in \mathbb{C} : 1 \leq |z| < \infty.$$

Note that Assumption 1 implies hereafter that x_0 is consistent, and Assumption 2 means that the slow and fast dynamics of system (1) are respectively detectable and observable (Dai, 1989).

The problem to be addressed consists in designing a state observer to provide an estimate $\hat{x}(k)$ of $x(k)$ such that:

1) The estimation error
$$e(k) := x(k) - \hat{x}(k) \tag{3}$$
converges to zero as $k \to \infty$ with a given decay rate ρ;
2) The estimation error dynamics satisfies a given H_2 performance with respect to process disturbances; and
3) The closed-loop admissibility of system (1) with
$$u(k) = -K\hat{x}(k)$$
is preserved, where $K \in \mathbb{R}^{m\times n}$ is a given admissibilizing state-feedback gain.

We end this section by introducing the following lemmas which are instrumental to derive the main results of this paper.

Lemma 2. Let $\rho \in (0, 1)$ be a given scalar. The system

$$x(k+1) = Ax(k),\ x \in \mathbb{R}^n, x(0) = x_0, \tag{4}$$

is asymptotically stable with a guaranteed decay rate ρ, i.e.:

$$\|x(k)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\ \rho^k\ \|x_0\|,\ \forall\ k \in \mathbb{N}^+,$$

if there exists a symmetric matrix $P \in \mathbb{R}^{n\times n}$ satisfying the following

$$\begin{bmatrix} -\rho P & PA \\ A'P & -\rho P \end{bmatrix} < 0, \tag{5}$$

where λ_1 and λ_2 are the smallest and largest eigenvalues of P, respectively.

Lemma 2 follows straightforwardly from Proposition 4.4 in (Duan and Yu, 2013).

Lemma 3. (Han et al., 2018) Let $\mathcal{W} \in \mathbb{R}^{l\times n}$ and $\mathcal{Y} \in \mathbb{R}^{n\times n}$ be given matrices, with $l \geq n$. There exists a matrix $\mathcal{X} \in \mathbb{R}^{n\times l}$ such that $\mathcal{X}\mathcal{W} = \mathcal{Y}$ if and only if

$$\text{rank}\left(\begin{bmatrix} \mathcal{W} \\ \mathcal{Y} \end{bmatrix}\right) = \text{rank}(\mathcal{Y}). \tag{6}$$

Moreover, the general solution of $\mathcal{X}\mathcal{W} = \mathcal{Y}$ is given by

$$\mathcal{X} = \mathcal{Y}\mathcal{W}^\sharp + \mathcal{Z}(I_l - \mathcal{W}\mathcal{W}^\sharp), \tag{7}$$

where $\mathcal{Z} \in \mathbb{R}^{n\times l}$ is an arbitrary matrix.

3. OBSERVER DESIGN

Notice from assumption 2-(i) that there always exist real matrices $T \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{n\times p}$ such that the following holds (Ben-Israel and Greville, 2003):

$$TE + RC = I_n. \tag{8}$$

Furthermore, the general solution of (8) can be obtained by means of Lemma 3 which yields:

$$[T \mid R] = \begin{bmatrix} E \\ C \end{bmatrix}^\sharp + Y\left(I_{n+p} - \begin{bmatrix} E \\ C \end{bmatrix}\begin{bmatrix} E \\ C \end{bmatrix}^\sharp\right). \tag{9}$$

with the matrix $Y \in \mathbb{R}^{n\times(n+p)}$ being arbitrary.

Taking into account that Y in (9) is a free matrix, we can assume without loss of generality there always exists a full rank T such that (8) holds as summarized in the following result.

Lemma 4. Let $E \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$ be two given matrices satisfying Assumption 2-(i). Then, there exists a solution $T \in \mathbb{R}^{n\times n}$, with $\text{rank}(T) = n$, and $R \in \mathbb{R}^{n\times p}$ as in (9) such that (8) holds.

Proof. Let $\Omega = \begin{bmatrix} E^T & C^T \end{bmatrix}^T$. Since Ω is full column rank, an admissible $\Omega^\sharp$ is a left inverse of Ω, i.e., $\Omega^\sharp\Omega = I_n$. Thus, a possible $\Omega^\sharp$ is given by

$$\Omega^\sharp = \left(\Omega^T\Omega\right)^{-1}\Omega^T = \left(\Omega^T\Omega\right)^{-1}\begin{bmatrix} E^T & C^T \end{bmatrix} \tag{10}$$

Hence, in view of (10), it follows that

$$I_{n+p} - \Omega\Omega^\sharp = I_{n+p} - \Omega\left(\Omega^T\Omega\right)^{-1}\begin{bmatrix} E^T & C^T \end{bmatrix} \tag{11}$$

Next, considering (10) and (11), the following can be readily derived from (9)

$$T = \left(I_n - Y\Omega\right)\left(\Omega^T\Omega\right)^{-1}E^T + Y_1 \tag{12}$$

$$R = \left(I_n - Y\Omega\right)\left(\Omega^T\Omega\right)^{-1}C^T + Y_2 \tag{13}$$

where $Y_1 \in \mathbb{R}^{n\times n}$ and $Y_2 \in \mathbb{R}^{n\times p}$ are arbitrary matrices such that $[Y_1\ Y_2] = Y$. In light of (12), there always exists a matrix Y_1 such that T is nonsingular. □

Then, we introduce an algebraic model transformation which yields a standard state-space representation of system (1) based on (8). To this end, pre-multiplying the first equation of (1) by T and taking (8) and the fact that $y(k) = Cx(k)$ into account yields:

$$\begin{aligned} TEx(k+1) &= TAx(k) + TBu(k) \\ (I_n - RC)x(k+1) &= TAx(k) + TBu(k) \\ x(k+1) &= TAx(k) + TBu(k) + RCx(k+1) \end{aligned}$$

leading to the following standard state-space representation

$$\begin{aligned} x(k+1) &= TAx(k) + TBu(k) + Ry(k+1) \\ y(k) &= Cx(k),\ x(0) = x_0 \end{aligned} \tag{14}$$

Notice that the detectability of (1) implies the detectability of (14) for a nonsingular matrix T; see, e.g., (Guo et al., 2019).

In view of the above developments, the following observer is proposed:

$$\begin{aligned} \hat{x}(k+1) &= TA\hat{x}(k) + TBu(k) + Ry(k+1) \\ &\qquad + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C\hat{x}(k),\ \hat{x}(0) = \hat{x}_0, \end{aligned} \tag{15}$$

where $\hat{x} \in \mathbb{R}^n$ is the observer state, $\hat{y}$ is an estimate of $y(k)$ and $L \in \mathbb{R}^{n\times p}$ is to be designed such that the estimation error $e(k)$ as defined in (3) converges to zero as $k \to \infty$.

Before introducing the main result of this section which establishes an LMI condition for designing the observer gain L as well as the matrices T and R, notice that the error dynamics can be easily derived from (3), (14) and (15) yielding:

$$e(k+1) = (TA - LC)e(k),\ e(0) = e_0 = x_0 - \hat{x}_0, \tag{16}$$

and consider the following auxiliary notation associated to the solution given in (9):

$$\begin{aligned} \begin{bmatrix} E \\ C \end{bmatrix}^\sharp &= [X_1\ X_2] \\ Y &= [Y_1\ Y_2] \\ W = I_{n+p} - \begin{bmatrix} E \\ C \end{bmatrix}\begin{bmatrix} E \\ C \end{bmatrix}^\sharp &= \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \end{aligned} \tag{17}$$

where $X_1 \in \mathbb{R}^{n\times n}$, $X_2 \in \mathbb{R}^{n\times p}$, $Y_1 \in \mathbb{R}^{n\times n}$, $Y_2 \in \mathbb{R}^{n\times p}$, $W_{11} \in \mathbb{R}^{n\times n}$, $W_{12} \in \mathbb{R}^{n\times p}$, $W_{21} \in \mathbb{R}^{p\times n}$ and $W_{22} \in \mathbb{R}^{p\times p}$.

In view of (9) and (17), notice that T and R can be cast as follows:

$$T = X_1 + Y_1W_{11} + Y_2W_{21} \tag{18}$$

$$R = X_2 + Y_1W_{12} + Y_2W_{22} \tag{19}$$

Theorem 1. Consider the error system in (16), satisfying Assumptions 1 and 2, the state observer in (15), and the error dynamics in (16). Let $\rho \in (0, 1)$ be a given scalar. Then, the error dynamics is asymptotically stable, with a guaranteed decay rate ρ, if there exist matrices $P = P^T \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{n\times n}$, $Y_{1Z} \in \mathbb{R}^{n\times n}$, $Y_{2Z} \in \mathbb{R}^{n\times p}$, $L_Z \in \mathbb{R}^{n\times p}$, such that the following LMI holds

$$\begin{bmatrix} \rho(P-Z-Z^T) & T_Z A - L_Z C \\ \star & -\rho P \end{bmatrix} < 0 \tag{20}$$

where

$$T_Z = ZX_1 + Y_{1Z}W_{11} + Y_{2Z}W_{21}. \tag{21}$$

In affirmative case, the matrix Z is nonsingular, the observer gain matrices are given by

$$L = Z^{-1}L_Z \tag{22}$$

$$T = X_1 + Z^{-1}\big(Y_{1Z}W_{11} + Y_{2Z}W_{21}\big) \tag{23}$$

$$R = X_2 + Z^{-1}\big(Y_{1Z}W_{12} + Y_{2Z}W_{22}\big) \tag{24}$$

and the error trajectory satisfies

$$\|e(k)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\, \rho^k \, \|e(0)\|, \ \forall \ k \geq 0, \tag{25}$$

where λ_1 and λ_2 are the smallest and largest eigenvalues of P, respectively.

Proof. Suppose that (20) is satisfied for some P, Z, L_Z, Y_{1Z} and Y_{2Z} and let $V(k) = e(k)^T Pe(k)$ be a Lyapunov function candidate.

Firstly, note from (20) that $Z + Z^T > P > 0$. Then, Z is nonsingular and $V(k) > 0$ for all $e(k) \neq 0$.

Next, taking into account that

$$P - Z - Z^T \geq -ZP^{-1}Z^T,$$

for any nonsingular Z, it follows from (20) and (22) that

$$\begin{bmatrix} -\rho Z P^{-1} Z^T & Z(TA - LC) \\ \star & -\rho P \end{bmatrix} < 0 \tag{26}$$

considering T as in (18) with

$$Y_1 = Z^{-1}Y_{1Z} \ \text{ and } \ Y_2 = Z^{-1}Y_{2Z}.$$

Then, pre- and post-multiplying (26) by

$$\text{diag}\{PZ^{-1}, I_n\} \ \text{ and } \ \text{diag}\{Z^{-T}P, I_n\},$$

respectively, leads to

$$\begin{bmatrix} -\rho P & P(TA - LC) \\ \star & -\rho P \end{bmatrix} < 0, \tag{27}$$

which from Lemma 2 implies that the error system (16) is asymptotically stable with a decay rate ρ. □

In the sequel, we provide a statement that if there is a solution to Theorem 1, then T as defined in (22) turns out to be nonsingular.

Proposition 1. Suppose there exist $P = P'$, Z, Y_{1Z}, Y_{2Z} and L_Z satisfying the LMI in (20). Then, there is a sufficiently small real number ϵ such that (20) holds for the same matrices $P = P'$, Z, Y_{2Z}, L_Z and Y_{1Z} replaced by

$$\tilde{Y}_{1Z} = Y_{1Z} + \epsilon I_n, \tag{28}$$

and the matrix

$$T_Z = T_Z(Z, \tilde{Y}_{1Z}, Y_{2Z}) = ZX_1 + \tilde{Y}_{1Z}W_{11} + Y_{2Z}W_{21}$$

is nonsingular.

Proof. Let the left-hand side of (20) be denoted as $\Omega(P, L_Z, Y_{1Z}, Y_{2Z}, Z)$. Then, it follows that

$$\Omega(P, L_Z, \tilde{Y}_{1Z}, Y_{2Z}, Z) = \Omega(P, L_Z, Y_{1Z}, Y_{2Z}, Z) + \epsilon\Delta\Omega, \tag{29}$$

where

$$\Delta\Omega = \begin{bmatrix} 0 & W_{11}A \\ \star & 0 \end{bmatrix},$$

and

$$T_Z(Z, \tilde{Y}_{1Z}, Y_{2Z}) = T_Z(Z, Y_{1Z}, Y_{2Z}) + \epsilon W_{11} \tag{30}$$

If $T_Z(Z, Y_{1Z}, Y_{2Z})$ is singular, in view of (29), (30) and the fact that $\Omega(P, L_Z, Y_{1Z}, Y_{2Z}, Z) < 0$, then there exists a sufficiently small ϵ such that $\Omega(P, L_Z, \tilde{Y}_{1Z}, Y_{2Z}, Z) < 0$ and $T_Z(Z, \tilde{Y}_{1Z}, Y_{2Z})$ is nonsingular. □

4. H_2 OBSERVER DESIGN

Suppose that system (1) is subject to an exogenous disturbance vector $w \in \mathbb{R}^g$, that is:

$$\begin{cases} Ex(k+1) = Ax(k) + Bu(k) + B_w w(k), \\ y(k) = Cx(k), \end{cases} \tag{31}$$

where $B_w \in \mathbb{R}^{n\times g}$. To ensure that the initial condition x_0 is consistent according to the first statement of Definition 1, we assume that $w(k)$ is a zero mean white noise sequence with identity power spectral density with $w(0) = 0$.

This section deals with the design of a state observer to deliver an estimate $\hat{x}(k)$ of the state vector $x(k)$ of system (31) in an H_2 sense with respect to a performance signal defined by a linear combination of the estimation error

$$e(k) := x(k) - \hat{x}(k),$$

which is represented by the output performance vector $s \in \mathbb{R}^h$ to be defined later in this section.

Then, pre-multiplying the dynamic equation of (31) by T and noting that

$$TEx(k+1) = x(k+1) - RCx(k+1)$$

leads to the following (standard) state-space representation

$$x(k+1) = TAx(k) + TBu(k) + Ry(k+1) + TB_w w(k) \tag{32}$$

Next, considering the state observer defined in (15) and the above system representation, we are interested in determining the matrices L, T and R such that

$$\|\mathcal{G}\|_2^2 \leq \mu \tag{33}$$

where μ is a given positive scalar and $\|\mathcal{G}\|_2$ denotes the H_2-norm of the estimation error system defined as follows:

$$\mathcal{G} : \begin{cases} e(k+1) = (TA - LC)e(k) + TB_w w(k) \\ s(k) = C_s e(k), \ e(0) = 0, \end{cases} \tag{34}$$

with $C_s \in \mathbb{R}^{h\times n}$.

The next theorem presents an LMI method for designing the observer defined in (15), which ensures a prescribed upper bound on the H_2-norm of system $\mathcal{G}$.

Theorem 2. Let μ be a given positive scalar. Consider the system in (31), satisfying Assumption 2, the state observer in (15), and the error dynamics in (34). Then, there exists an observer such that $\|\mathcal{G}\|_2^2 < \mu$ if there are matrices $P = P^T \in \mathbb{R}^{n\times n}$, $M = M^T \in \mathbb{R}^{h\times h}$, $Z \in \mathbb{R}^{n\times n}$, $Y_{1Z} \in \mathbb{R}^{n\times n}$, $Y_{2Z} \in \mathbb{R}^{n\times p}$ and $L_Z \in \mathbb{R}^{n\times p}$ such that the following LMIs hold

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$$\mu - \text{trace}\{M\} > 0 \tag{35}$$

$$\begin{bmatrix} M & C_s \\ \star & Z + Z^T - P \end{bmatrix} > 0 \tag{36}$$

$$\begin{bmatrix} -P & \star & \star \\ (T_Z A - L_Z C)^T & P - Z - Z^T & \star \\ (T_Z B_w)^T & 0 & -I_g \end{bmatrix} < 0 \tag{37}$$

where T_Z is as in (21). Moreover, the observer gains are as given in (22), (23) and (24).

Proof. From the Extended H_2-norm computation result of (De Oliveira et al., 2002, Theorem 1), with

$$Z = \mathcal{G}^{-1} \text{ and } P = \mathcal{G}^{-1}\mathcal{P}\mathcal{G}^{-T},$$

the system in (34) is asymptotically stable and $\|\mathcal{G}\|_2^2 < \mu$, if there exist $P = P^T \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{n\times n}$, and $M = M^T \in \mathbb{R}^{h\times h}$ such that the following matrix inequalities are satisfied

$$\mu - \text{trace}\{M\} > 0 \tag{38}$$

$$\begin{bmatrix} M & C_s \\ C_s^T & Z + Z^T - P \end{bmatrix} > 0 \tag{39}$$

$$\begin{bmatrix} -P & \star & \star \\ (TA - LC)^T Z & P - Z - Z^T & \star \\ (TB_w)^T Z & 0 & -I_g \end{bmatrix} < 0 \tag{40}$$

Next, letting

$$\begin{aligned} ZT &= T_Z = Z(X_1 + Y_1 W_{11} + Y_2 W_{21}) \\ ZY_1 &= Y_{1Z} \\ ZY_2 &= Y_{2Z} \\ ZL &= L_Z \end{aligned}$$

the conditions in (38), (39) and (40) are equivalent to (35), (36) and (37), respectively. The expression of R in (24) follows straightforwardly from (19), which completes the proof. □

Notice without loss of generality that T_Z (and thus T) can be be assumed nonsingular following the arguments stated in Proposition 1.

Remark 1. When the matrices T and R are given *a priori*, notice that Theorem 2 provides a necessary and sufficient condition to ensure that $\|\mathcal{G}\|_2^2 \le \mu$. However, as shown in Section 6, considering the matrices T and R as decision variables will either provide the same result or lead to a smaller upper bound on $\|\mathcal{G}\|_2^2 \le \mu$.

5. SEPARATION PRINCIPLE

Let the system defined in (1), with the control law

$$u(k) = -K\hat{x}(k),\ K \in \mathbb{R}^{m\times n}, \tag{41}$$

and consider the following assumption

Assumption 3. There exists a real matrix $K \in \mathbb{R}^{m\times n}$ such that system (1), with the following control law

$$u(k) = -Kx(k), \tag{42}$$

is admissible.

The above assumption implies that the slow dynamics of (1) is stabilizable, i.e.,

$$\text{rank}\left(\begin{bmatrix}(zE - A) & B\end{bmatrix}\right) = n,\ \forall\, z \in \mathbb{C} : 1 \le |z| < \infty,$$

and its fast dynamics is *causal*-controllable (Belov et al., 2018), i.e.,

$$\text{rank}\left(\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix}\right) = n + r.$$

Then, we show in the sequel that the control law in (41) under Assumption 3 preserves the admissibility of the closed-loop system. More specifically, we demonstrate a separation principle property for the following augmented (closed-loop) system:

$$\begin{cases} Ex(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k), \\ \hat{x}(k+1) &= (TA - LC)\hat{x}(k) + TBu(k) \\ & \qquad + Ry(k+1) + Ly(k) \\ u(k) &= -K\hat{x}(k) \end{cases} \tag{43}$$

meaning that the control gain K and the observer matrices L, T and R can be designed independently provided that (8) holds.

In view of (16) and (43), notice that the closed-loop system can be described in terms of $x(k)$ and $e(k)$ via the following equation:

$$Ex(k+1) = (A - BK)x(k) + BK\big(x(k) - \hat{x}(k)\big) \tag{44}$$

which together with (16), lead to the following descriptor representation:

$$\begin{bmatrix} I_n & 0_n \\ 0_n & E \end{bmatrix} \begin{bmatrix} e(k+1) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} TA - LC & 0 \\ BK & (A - BK) \end{bmatrix} \begin{bmatrix} e(k) \\ x(k) \end{bmatrix} \tag{45}$$

By noting that the dynamic matrix of the augmented system (45) is lower block triangular, then the eigenvalues of the estimation error sub-system can be freely assigned. Furthermore, the system closed-loop dynamics defined in (44) is admissible from Assumption 3 and the fact that $e(k) = x(k) - \hat{x}(k)$ vanishes to zero as $k \to \infty$ assuming that $(TA - LC)$ is Schur stable.

The above developments are summarized in the following result.

Theorem 3. Let $K \in \mathbb{R}^{m\times n}$ be a given matrix such that Assumption 3 holds. Then, system (43) is admissible if and only if the matrix $(TA - LC)$ is Schur stable.

The proof of above theorem is straightforward from the system representation in (45) and is omitted for brevity.

6. NUMERICAL EXAMPLES

The following numerical examples show the potentials of the proposed approach for observer and observer-based output feedback control design for linear discrete-time systems.

6.1 Output feedback design

Consider the descriptor system defined in (1) with the following matrices:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix},\quad A = \begin{bmatrix} -0.25 & 0.00 & 0.00 \\ -0.50 & 0.50 & 2.00 \\ 0.75 & -1.00 & -1.50 \end{bmatrix},$$

$$B = [0\ 0\ 1]^T,\quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{46}$$

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which has adapted from (Belov and Andrianova, 2019, Example 5.1).

The above matrices imply that system (1) in open-loop is unstable, but it satisfies Assumptions 2 and 3. Hence, in this example, we are interested in determining an admissibilizing output feedback control law as in (41). To this end, we design a state-feedback control law following the approach of (Masubuchi and Ohta, 2013) with a guaranteed decay rate of 0.9 leading to:

$$K = -\left[0.710\ 1.000\ 2.230\right] \quad (47)$$

Then, Theorem 1, with $\rho = 0.2$, is applied to determine the state observer (i.e., the matrices L, T and R) which yields

$$L = \begin{bmatrix} -0.1250 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.6311 & -0.6311 \end{bmatrix}, \quad R = \begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 1.0 \\ -1.0 & 0.0 \end{bmatrix}, \quad (48)$$

$$T = \begin{bmatrix} 0.5000 & 0.0000 & 0.0000 \\ 0.0000 & 1\times 10^{-12} & 0.0000 \\ -1.0000 & 0.7377 & 1.0000 \end{bmatrix}.$$

Figures 1, 2 and 3 show respectively the states of the closed-loop system with $u(k) = -K\hat{x}(k)$ (i.e., output feedback) and $u(k) = -Kx(k)$ (i.e., state-feedback) as well as the estimation error trajectory $e(k) = x(k) - \hat{x}(k)$ considering the output feedback controller and an admissible initial condition given by

$$x(0) = \left[1 \quad 5 \quad -1\right]^T$$

with $\hat{x}(0) = 0$. Notice the fast response achieved by the proposed observer and the good performance achieved by the output feedback control law $u(k) = -K\hat{x}(k)$ when compared to the state feedback controller $u(k) = -Kx(k)$ demonstrating the efficiency of the proposed methodology.

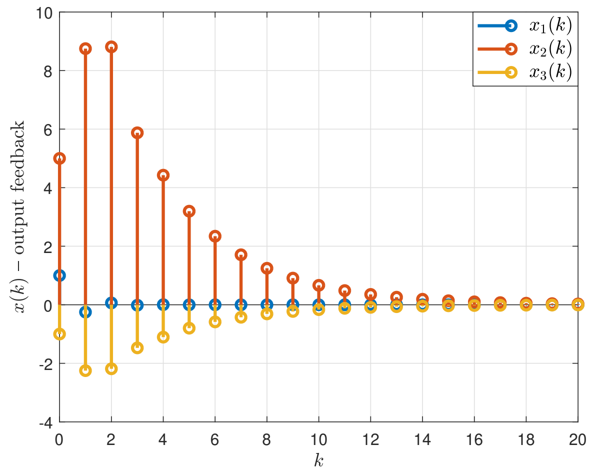

Figure 1. State trajectories of the closed-loop system with $u(k) = -K\hat{x}(k)$ (output feedback controller).

6.2 H_2 observer design

Consider the following system adapted from (Wang et al., 2012, Example 1) and suppose that $w(k)$ is a zero mean white noise:

$$\begin{cases} Ex(k+1) = Ax(k) + B_w w(k) \\ y(k) = Cx(k) \end{cases} \quad (49)$$

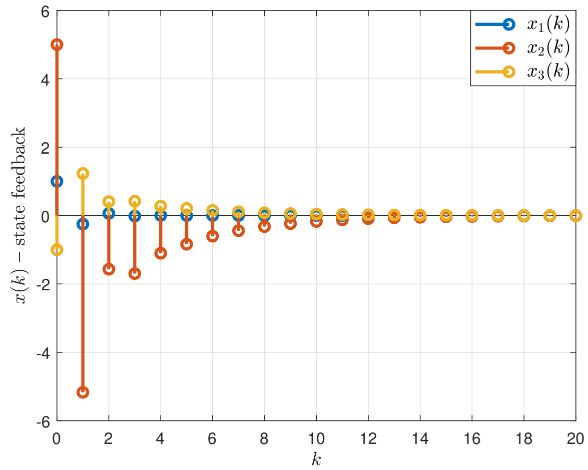

Figure 2. State trajectories of the closed-loop system with $u(k) = -Kx(k)$ (state feedback controller).

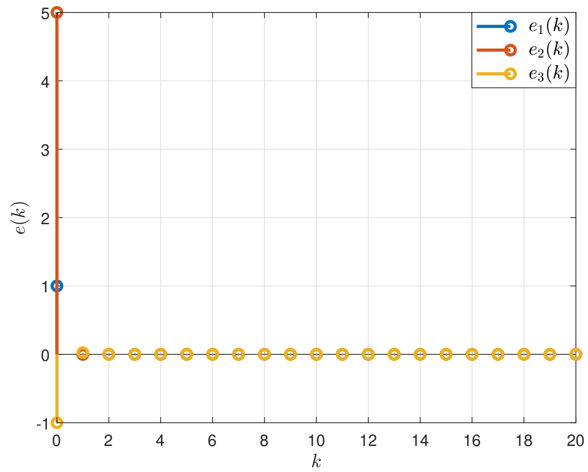

Figure 3. Estimation error trajectory for $e(0) = x(0)$.

where

$$E = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.153 & 0.045 & 0.069 \\ 0.156 & 0.252 & 0.156 \\ 0.135 & -0.171 & -0.635 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 1.0 \\ 0.5 \\ 0.5 \end{bmatrix}, \quad C = \left[0\ 0\ 1\right].$$

Note that Assumption 1 holds for the descriptor system defined by the above matrices. This example aims at designing an observer as in (15) which minimizes $\|\mathcal{G}\|_2$, where $\mathcal{G}$ is the transfer function matrix of the estimation error system defined in (34) with

$$C_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

To this end, we apply the optimization problem:

$$\min_{P,M,Z,L_Z,Y_{1Z},Y_{2Z},\mu} \mu \ \text{ subject to (35)-(37)}$$

leading to the following results:

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$$L = \begin{bmatrix} -4.6023 \times 10^{-1} \\ 5.7175 \times 10^{-1} \\ 1.6099 \times 10^{-4} \end{bmatrix}, \quad (50)$$
$$T = \begin{bmatrix} 3.1764 \times 10^{-1} & -3.1764 \times 10^{-1} & 6.8236 \times 10^{-1} \\ 9.0356 \times 10^{-1} & -4.0356 \times 10^{-1} & -9.0356 \times 10^{-1} \\ 1.7753 \times 10^{-5} & -1.7753 \times 10^{-5} & -1.7753 \times 10^{-5} \end{bmatrix},$$
$$R = \begin{bmatrix} 0.0 \\ -0.5 \\ 1.0 \end{bmatrix}, \quad \|\mathcal{G}\|_2 = \sqrt{\mu} = 0.5923.$$

On the other hand, if we consider that the matrices T and R are fixed and given by

$$T = \begin{bmatrix} -7.0 & 7.0 & 8.0 \\ -7.0 & 7.5 & 7.0 \\ -7.0 & 7.0 & 7.0 \end{bmatrix}, \quad R = \begin{bmatrix} 0.0 \\ -0.5 \\ 1.0 \end{bmatrix},$$

we obtain $\|\mathcal{G}\|_2 = 1.0094$ which is 70% larger than the proposed result.

7. CONCLUDING REMARKS

This work has proposed sufficient conditions for observer and observer-based output feedback design for linear discrete-time descriptor systems. Precisely, the equality constraint associated to the observability condition of the system fast dynamics is applied to derive a state observer having a standard state-space representation with a guaranteeing convergence rate. Contrasting with similar results in specialized literature, we consider arbitrary matrices linked to the solution of the fast dynamics observability constraint as decision variables yielding less conservative results for H_2 observer-based filtering design when comparing to the results utilizing fixed arbitrary matrices. Moreover, a separation principle has been demonstrated, with some mild assumptions, showing that the observer and the state-feedback can be independently designed. Numerical examples have clearly demonstrated the potentials of the proposed approach as a tool to observer and observer-based output feedback design.

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