Pareto-Nash Equilibrium in Multiobjective Formation Control Problems

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Abstract: The formation control of teams of agents constitutes an important class of multiagent control systems. We combine control, graph and game theories to address a particular formation control problem – the formation tracking problem – in the context of the multiobjective noncooperative game theory. The solution concept proposed for the problem is the Pareto-Nash equilibrium, which is at the same time a Nash equilibrium (stable, self-enforcing) and, with respect to the Nash equilibrium, Pareto-optimal for each agent in the formation. A numerical example illustrates the main characteristics and contributions brought by the approach and solution concept proposed.

Keywords: Multiagent Systems, Formation Control, LQ Control, Graph Theory, Game Theory

1. INTRODUCTION

Formation control problems (FCPs, henceforth) have been extensively investigated in the last few decades and constitute a consolidated research area in control systems engineering (Cao et al. (2013), Lewis et al. (2014), Chen and Ren (2019)), with practical applications in the coordination of aerial vehicles, mobile robots and sensor networks, among others. Formation control explores the fact that a team of agents working as a group often exceeds the performance of the same team of agents working individually.

In formation tracking control, in particular, a team of agents must reach some predefined geometrical configuration while following a prescribed team reference (Wang ans Slotine (2006), Ni and Cheng (2010)). In the leader-follower approach for formation tracking, one agent plays the role of leader and the other agents act as followers. The leader moves along a predefined trajectory and all the agents must keep prescribed distances from each other, which produces the geometrical and trajectory configurations of interest.

Formation tracking problems have been addressed using a rich combination of systems theory, algebraic graph theory, and game theory. In Gu (2008), the formation control of mobile robots with double integrator dynamics is modeled as a non-cooperative graphical game. The solution of the FCP is characterized as an open-loop Nash equilibrium (Nash (1950a), Başar and Olsder (1999)). The same problem is addressed in Shamsi et al. (2011), Aghajani and Doustmohammadi (2011) and Horevicz and Ferreira (2021). Shamsi et al. (2011) provides a closed-loop Nash equilibrium solution for a particular time varying formation. Aghajani and Doustmohammadi (2011) proposes a cooperative approach centered in the concept of Pareto-optimal solution (Miettinen (1998)). Horevicz and Ferreira (2021) shows the superiority of the (cooperative) Nash bargaining solution (Nash (1950b)) over the (non-cooperative) Nash equilibrium solution for the FCP. Bardhan and Ghose (2018) proposes a negotiation procedure involving the Nash barganing solution and the Nash equilibrium solution, applied to a rendezvous problem of a team of unmanned aerial vehicles (UAVs).

Although the existing approaches for the FCP assume that a single objective is associated to each agent, nothing precludes us to consider the FCP in the context of multiobjective games (Shapley (1959)). The existence of Pareto equilibria in multiobjective games is discussed, among others, in Wang (1993), and rely on the quasi-convexity of the functionals and the compactness of the strategies space. An investigation about bargaining equilibria in multiobjective games, focused on the Pareto-Nash equilibrium, is presented in Wang and Yang (2021).

In this paper we formulate and solve the FCP in the context of multiobjective games. The Pareto-Nash solution that we propose is a Nash equilibrium (stable, self-enforcing) with the additional property of being Pareto-optimal for each agent with respect to the Nash equilibrium, in contrast to the Nash bargaining solution (Horevicz and Ferreira (2021)), which is Pareto-optimal for the grand coalition of the agents.

The paper is organized as follows. In Section 2, we introduce the dynamics of the agents, the elements of graph theory used to describe their interconnections, and the affine-quadratic formulation of the FCP. In Section 3 and 4, respectively, we characterize the Pareto and Nash equilibria, and in Section 5 we introduce the concept of Pareto-Nash equilibrium for the FCP. In Section 6, a numerical example illustrates the main characteristics and contributions brought by the game-theoretic approach proposed. Finally, in Section 7 we present our conclusions and a topic for future research.

2. PROBLEM FORMULATION

2.1 Notation and Conventions

Throughout the article, $\mathbb{R}^n$, $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the n-dimensional real space and the sets of n-dimensional vectors

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with nonnegative and positive coordinates, respectively. If $x, y \in \mathbb{R}^n$, then $x \leq y$ (resp., $x < y$) means that $x_i \leq y_i$ for $i = 1, 2, \ldots, n$ (resp., $x_i < y_i$ for $i = 1, 2, \ldots, n$). The space of all real matrices with m rown and n columns is denoted by $\mathbb{R}^{m\times n}$. The shorthand notation $M = \{m_{ij}\}$ is used to define $M \in \mathbb{R}^{m\times n}$; $M = M^T \geq 0$ (resp., $M = M^T > 0$) means that M is a symmetric positive semidefinite (resp., positive definite) matrix. Quadratic forms $x^T Mx$, $M = M^T \in \mathbb{R}^{n\times n}$, are written as $\|x\|_M^2$. The Kronecker product of matrices A and B of arbitrary sizes is the matrix defined by $A \otimes B = \{a_{ij}B\}$. If $\mathscr{A}$ is a set, then $|\mathscr{A}|$ is the number of elements (cardinality) of $\mathscr{A}$.

2.2 System Dynamics

We consider linear controllable systems with double integrator dynamics, which is a standard mass-force model for wide variety of autonomous vehicles. See, for example, Cao et al. (2011) and Wu et al. (2021). Let $\mathscr{N} := \{1, 2, \ldots, N\}$ be a set of N identical systems – agents:

$$\dot{x}_i = \Phi x_i + \Omega u_i, \quad i \in \mathscr{N}, \tag{1}$$

where $x_i = (q_i, \dot{q}_i) \in \mathbb{R}^{2n}$ is the state vector, described by position (q_i) and velocity ($\dot{q}_i$) coordinates, and $u_i : [0, T] \to \mathbb{R}^n$ is, by assumption, for some finite time horizon T, the piecewise continuous control vector of agent $i \in \mathscr{N}$. The state and control matrices are

$$\Phi = \begin{bmatrix} O_n & I_n \\ O_n & O_n \end{bmatrix} \quad \text{and} \quad \Omega = \begin{bmatrix} O_n \\ I_n \end{bmatrix},$$

where O_n and I_n are the n-order zero and identity matrices. Defining $x = [x_1^T \; x_2^T \; \cdots \; x_N^T]^T$, $u = [u_1^T \; u_2^T \; \cdots \; u_N^T]^T$, $A = I_N \otimes \Phi$ and $B_i = [0 \;\; 0 \; \cdots \; 1 \; \cdots \; 0]^T \otimes \Omega$, we obtain the global state equation

$$\begin{aligned} \dot{x} &= Ax + \sum_{i\in\mathscr{N}} B_i u_i \\ &= Ax + Bu, \end{aligned} \tag{2}$$

where $B := [B_1 \;\; B_2 \; \cdots \; B_N]$. The reference trajectory r_i for state x_i ($i \in \mathscr{N}$) is constrained to its dynamics, that is,

$$\dot{r}_i = \Phi r_i + \Omega \rho_i,$$

where ρ_i is the reference control input that produces r_i. Similarly, the global reference state equation is

$$\dot{r} = Ar + B\rho. \tag{3}$$

2.3 Graph Representation

As in Gu (2008) and Horevicz and Ferreira (2021), algebraic graph theory is used to obtain an equivalent affine-quadratic FCP. The interconnections of a set of agents is represented by a *graph* $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V}$ and $\mathscr{E}$ are the sets of *vertices* and *edges* of $\mathscr{G}$. Vertices are associated to agents and edges to their interconnections. A *directed* graph $\mathscr{G}$ consists of a vertex set $\mathscr{V} = \{v_1, v_2, \ldots, v_N\}$ and an ordered set of edges $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$. By assumption, $\mathscr{G}$ does not have *self-loops*, that is, if $(v_i, v_j) \in \mathscr{E}$ then $v_i \neq v_j$. A directed graph $\mathscr{G}$ is *connected* if there exists a *path in* $\mathscr{E}$ (a sequence of edges in $\mathscr{E}$) from v_i to v_j for any two vertices $v_i, v_j \in \mathscr{V}$. We assume that the team of agents is represented by a connected graph $\mathscr{G}$.

The *incidence matrix* of a directed graph $\mathscr{G}$ is the matrix $G \in \mathbb{R}^{N\times|\mathscr{E}|}$ whose rows and columns represent the vertices and edges of $\mathscr{G}$, respectively. Specifically, $G = \{g_{\nu\varepsilon}\}$, and $g_{\nu\varepsilon} = 1$, if ν is the *head* of the edge ε, $g_{\nu\varepsilon} = -1$ if ν is the *tail* of the edge ε, and $g_{\nu\varepsilon} = 0$, otherwise.

2.4 Formation Control

The formation error of the system represented by $\mathscr{G}$ is (Gu (2008), Horevicz and Ferreira (2021)):

$$\sum_{(i,j)\in\mathscr{E}} w_{ij} \|x_i - x_j - d_{ij}\| = \|x - r\|_{\bar{L}}^2, \tag{4}$$

where $d_{ij} = r_i - r_j$ is the desired distance between neighbour agents (vertices) $i, j \in \mathscr{N}$, $\bar{L} = L \otimes I_{2n}$, where $L = GWG^T$ is the *Laplacian matrix* of $\mathscr{G}$ ($L = L^T \geq 0$), and $W \in \mathbb{R}^{|\mathscr{E}|\times|\mathscr{E}|}$, $W = \{w_{ij}\}$, is a diagonal matrix of weights $w_{ij} \geq 0$.

Let $W_i, W_{if} \in \mathbb{R}^{|\mathscr{E}|\times|\mathscr{E}|}$, $i \in \mathscr{N}$, be diagonal matrices such that $w_{ij} > 0$ if $(i, j) \in \mathscr{E}$ and $w_{ij} = 0$, otherwise. Let $L_i = GW_iG^T$ and $L_{if} = GW_{if}G^T$ be their corresponding Laplacian matrices. Then to each agent is associated a convex quadratic cost functional

$$\begin{aligned} J_i(u) = &\frac{1}{2}\|x(T) - r(T)\|_{Q_{if}}^2 + \\ &+ \frac{1}{2}\int_0^T \left[\|x(t) - r(t)\|_{Q_i}^2 + \|u_i(t)\|_{R_i}^2\right] dt, \quad i \in \mathscr{N}, \end{aligned} \tag{5}$$

where the matrices $Q_{if} = L_{if} \otimes I_{2n}$ ($Q_{if} = Q_{if}^T \geq 0$) and $Q_i = L_i \otimes I_{2n}$ ($Q_i = Q_i^T \geq 0$) weigh the formation errors, and the matrix $R_i = R_i^T > 0$ weighs the control effort. The agents are also required to follow a trajectory r_ℓ assigned to some leader agent $\ell \in \mathscr{N}$. The cost functional of the leader agent is

$$\begin{aligned} J_\ell(u) = &\frac{1}{2}\left[\|x(T) - r(T)\|_{Q_{\ell f}}^2 + \|x_\ell(T) - r_\ell(T)\|_{K_{\ell f}}^2\right] + \\ &+ \frac{1}{2}\int_0^T \left[\|x(t) - r(t)\|_{Q_\ell}^2 + \|x_\ell(t) - r_\ell(t)\|_{K_\ell}^2 + \right. \\ &\left. + \|u_\ell(t)\|_{R_\ell}^2\right] dt, \end{aligned} \tag{6}$$

where $K_{\ell f} = K_{\ell f}^T \geq 0$ and $K_\ell = K_\ell^T \geq 0$ are weighting matrices for tracking errors. The cost functional of the leader agent ℓ can be written in the form (5) substituing $Q_{\ell f}$ and Q_ℓ with

$$Q_{\ell f} + \mathrm{diag}\,(O_{2n}, \ldots, O_{2n}, K_{\ell f}, O_{2n}, \ldots, O_{2n})$$

and $Q_\ell + \mathrm{diag}\,(O_{2n}, \ldots, O_{2n}, K_\ell, O_{2n}, \ldots, O_{2n})$, respectively. The FCP consists in solving, in some sense, the optimization problem

$$\min_u \; J(u) = (J_1(u), J_2(u), \ldots, J_N(u)) \;\; \text{subject to} \;\; (2), \tag{7}$$

where $u = (u_1, u_2, \ldots, u_N)$. In the next sections we characterize three game-theoretic equilibrium solutions for the FCP: Pareto, Nash and Pareto-Nash equilibria.

3. PARETO EQUILIBRIUM

The *Pareto equilibrium* is a cooperative solution concept for the set (grand coalition) of agents of the FCP. Under a Pareto equilibrium, any decrease in the cost functional of any agent is accompanied by an increase in the cost functional of at least one of the other agents. Let $J(u) = (J_1(u), J_2(u), \ldots, J_N(u))$ be the cost vector associated to the FCP. Then a solution $u^\star = (u_1^\star, u_2^\star, \ldots, u_N^\star)$ is *Pareto-optimal* if there exists no other solution u such that $J(u) \leq J(u^\star)$ with $J_i(u) < J_i(u^\star)$ for at least one $i \in \mathscr{N}$. A solution $u^\star$ is *weakly Pareto-optimal* if there exists no other solution u such that $J(u) < J(u^\star)$. Every Pareto-optimal solution is also weakly Pareto-optimal. Pareto-optimal solutions can be determined by using scalarization methods (Miettinen (1998)). Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be *weights* assigned to the costs $J_1, J_2, \ldots, J_N$. The domain of the weighting vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{R}^N$ is, without loss of generality,

$$\Lambda = \{\lambda \in \mathbb{R}_+^N \; : \; \sum_{i\in\mathscr{N}} \lambda_i = 1\}.$$

A weighted-sum quadratic cost functional J^λ is associated to each $\lambda \in \Lambda$:

$$J^\lambda(u) = \sum_{i=1}^N \lambda_i J_i(u) = \frac{1}{2}\|x(T) - r(T)\|^2_{Q_{f,\lambda}} + \\ + \frac{1}{2}\int_0^T \left\{\|x(t) - r(t)\|^2_{Q_\lambda} + \|u(t)\|^2_{R_\lambda}\right\} dt$$

where

$$Q_{f,\lambda} = \sum_{i\in\mathscr{N}} \lambda_i Q_{if}, \quad Q_\lambda = \sum_{i\in\mathscr{N}} \lambda_i Q_i$$

and $R_\lambda = \mathrm{diag}(\lambda_1 R_1, \lambda_2 R_2, \ldots, \lambda_N R_N)$. It is possible to show (Miettinen (1998)) that if u^λ, $\lambda \in \Lambda$, is an optimal solution of the convex optimization problem

$$\min_u J^\lambda(u) \text{ subject to (2)}, \tag{8}$$

then u^λ is a weakly Pareto-optimal solution of the FCP. If, in addition, (i) u^λ is the unique solution of problem (8), or (ii) $\lambda \in \mathbb{R}^N_{++}$, then u^λ is a Pareto-optimal solution of the FCP. The convexity of problem (8) guarantees that every Pareto-optimal solution of the FCP is an optimal solution of (8) for some $\lambda \in \Lambda$ (Miettinen (1998)).

Theorem 1. (Pareto Equilibrium). Given $\lambda \in \Lambda \cap \mathbb{R}^N_{++}$, assume that P and ξ are solutions for the differential equations

$$\dot{P}(t) = -P(t)A - A^T P(t) - Q_\lambda + \\ + P(t)\sum_{j\in\mathscr{N}} \lambda_j^{-1} S_j P(t), \quad P(T) = Q_{f,\lambda},$$

where $S_j = B_j R_j^{-1} B_j^T$, and

$$\dot{\xi}(t) = \left[P(t)\sum_{j\in\mathscr{N}} \lambda_j^{-1} S_j - A^T\right]\xi(t) - \\ - Q_\lambda r(t), \quad \xi(T) = Q_{f,\lambda} r(T).$$

Then $u^\lambda = (u_1^\lambda, u_2^\lambda, \ldots, u_N^\lambda)$ defined by

$$u_i^\lambda(t) = -\lambda_i^{-1} R_i^{-1} B_i^T [P(t)x(t) + \xi(t)], \quad i \in \mathscr{N}, \tag{9}$$

is a Pareto-optimal solution of problem (8).

Proof. The proof follows from the general solution of linear-quadratic tracking problems (Lewis and Syrmos (1995), pg. 217) particularized for the structures of the matrices R_λ (block-diagonal) and B (Horevicz and Ferreira (2021)). □

See Lewis and Syrmos (1995), Chapter 4, for a comprehensive discussion (which includes the stability of the closed-loop system) of the classical tracking problem and other LQR extensions. The Nash bargaining solution (Nash (1950b)) for the FCP proposed in Horevicz and Ferreira (2021) corresponds to affine state feedback controls (9) for a specific $\lambda \in \Lambda$ that minimizes the weighted-sum cost functional $J^\lambda(u)$.

4. NASH EQUILIBRIUM

The FCP can be modeled as a noncooperative nonzero sum graphical game represented by the tuple

$$\Gamma = \{\mathscr{G}, (J_i)_{i\in\mathscr{N}}, (u_i)_{i\in\mathscr{N}}\}.$$

The agents are assumed to be rational, solely aiming at minimizing their own cost functionals (5) subject to the state equation (2); the tuple Γ is assumed to be *common knowledge* among the players (Maschler et al. (2013)). A *strategic profile* $u^{\mathrm{NE}} = (u_1^{\mathrm{NE}}, u_2^{\mathrm{NE}}, \ldots, u_N^{\mathrm{NE}})$ is a *Nash equilibrium* (NE, henceforth) of Γ if

$$J_i(u_i^{\mathrm{NE}}, u_{-i}^{\mathrm{NE}}) \leq J_i(u_i, u_{-i}^{\mathrm{NE}})$$

for all controls u_i and all $i \in \mathscr{N}$, where u_{-i}^{NE} is the *deleted* profile from u^{NE} which does not include u_i^{NE}. Equivalently, $u^{\mathrm{NE}} = (u_i^{\mathrm{NE}}, u_{-i}^{\mathrm{NE}})$ is a Nash equilibrium of Γ if u_i^{NE} is a *best response* of the agent i when the remaining $N-1$ agents adopt the deleted profile u_{-i}^{NE}. Under a Nash equilibrium, there is no incentive for the agent i to unilaterally deviate from u_i^{NE} if all the other agents stick to u_{-i}^{NE}. For the characterization of Nash equilibria of the FCP stated in Theorem 2, we introduce the error vector $\delta = x - r$ and, using (3), rewrite the FCP as a quadratic regulator problem with affine dynamics:

$$\dot{\delta} = A\delta + Bu + c, \quad c = -B\rho. \tag{10}$$

Theorem 2. (Nash Equilibrium). Assume that for Γ there exists a solution set $\{P_i \; : \; i \in \mathscr{N}\}$ for the coupled Riccati differential equations

$$\dot{P}_i = -P_i(t)A - A^T P_i(t) - Q_i + \\ + P_i(t)\sum_{j\in\mathscr{N}} S_{ij} P_j(t), \quad P_i(T) = Q_{if},$$

where $S_{ij} = B_j R_i^{-1} B_j^T$. Then Γ admits an open loop Nash equilibrium $u^{\mathrm{NE}} = (u_1^{\mathrm{NE}}, u_2^{\mathrm{NE}}, \ldots, u_2^{\mathrm{NE}})$ described by

$$u_i^{\mathrm{NE}}(t) = -R_i^{-1} B_i^T [P_i(t)\delta(t) + \xi_i(t)], \quad i \in \mathscr{N}, \tag{11}$$

where $\{\xi_i \; : \; i \in \mathscr{N}\}$ solve uniquely the set of differential equations

$$\dot{\xi}_i = -A^T \xi_i(t) + P_i(t)\sum_{j\in\mathscr{N}} S_{ij}\xi_j(t) + P_i(t)B\rho(t), \quad \xi_i(T) = 0,$$

and δ solves (10) for the controls (11) and $\delta(0) = x(0) - r(0)$.

Proof. See Başar and Olsder (1999), Theorem 6.12. □

Conditions for the solution of coupled Riccati equations parameterized by the time horizon T are discussed in Başar and Olsder (1999) and Engwerda (2005). A closed-loop NE solution for the FCP can be also derived from Başar and Olsder (1999), Chapter 6.

5. PARETO-NASH EQUILIBRIUM

The Nash equilibrium is an stable (self-enforcing) equilibrium concept. When reached, it will not be breached even in the absence of binding agreements. However, a Nash equilibrium is not necessarily Pareto-optimal (Maschler et al. (2013)). On the other hand, as the NE is determined by a given definition of cost functionals, the agents may act strategically by choosing a cost functional which provides him (her), for example, the best compromise between tracking the reference trajectory and spending energy for that. Thus, it is natural to introduce a set $\mathscr{M}_i = \{1, 2, \ldots, M_i\}$ of convex quadratic cost functionals J_{ij} ($j \in \mathscr{M}_i$) for each agent $i \in \mathscr{N}$ and consider the multiobjective noncooperative (in the sense that each agent solely aims at minimizing his/her own cost functionals) graphical game

$$\Gamma_M = \{\mathscr{G}, (J_{ij})_{i\in\mathscr{N}, j\in\mathscr{M}_i}, (u_i)_{i\in\mathscr{N}}\}.$$

The weighted-sum cost functional of each agent is

$$J_i^{\lambda_i}(u) = \sum_{j\in\mathscr{M}_i} \lambda_{ij} J_{ij}(u), \quad i \in \mathscr{N},$$

where (with some abuse of notation) $\lambda_i = [\lambda_{i1}\ \lambda_{i2}\ \cdots\ \lambda_{iM_i}]^T$ and

$$\lambda_i \in \Lambda_i = \{\lambda_i \in \mathbb{R}_+^{M_i}\ :\ \sum_{j \in \mathscr{M}_i} \lambda_{ij} = 1\}, \quad i \in \mathscr{N}.$$

A strategy $\bar{u}_i$ of agent $i \in \mathscr{N}$ is Pareto-optimal (resp., weakly Pareto-optimal) with respect to the profile $\bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N)$ if there is no other u_i such that $J_{ij}(u_i, \bar{u}_{-i}) \leq J_{ij}(\bar{u}_i, \bar{u}_{-i})$ for all $j \in \mathscr{M}_i$ and $J_{ij}(u_i, \bar{u}_{-i}) < J_{ij}(\bar{u}_i, \bar{u}_{-i})$ for at least one $j \in \mathscr{M}_i$ (resp., $J_{ij}(u_i, \bar{u}_{-i}) < J_{ij}(\bar{u}_i, \bar{u}_{-i})$ for all $j \in \mathscr{M}_i$). A strategic profile $\bar{u}$ is a Pareto-optimal (resp., weakly Pareto-optimal) equilibrium of Γ_M if $\bar{u}_i$ is Pareto-optimal (resp., weakly Pareto-optimal) with respect to $\bar{u}$ for all $i \in \mathscr{N}$ (Wang (1993), Yuan and Tarafdar (1996)). Given weighting vectors $\lambda_i \in \Lambda_i,\ i \in \mathscr{N}$, a strategic profile $u^{\mathrm{NE}} = (u_i^{\mathrm{NE}}, u_{-i}^{\mathrm{NE}})$ is a Nash equilibrium of Γ_M if

$$J_i^{\lambda_i}(u_i^{\mathrm{NE}}, u_{-i}^{\mathrm{NE}}) \leq J_i^{\lambda_i}(u_i, u_{-i}^{\mathrm{NE}})$$

for all controls u_i and all $i \in \mathscr{N}$. The later inequalities imply that u^{NE} as a weakly Pareto-Nash equilibrium of Γ_M, since u_i^{NE} is weakly Pareto-optimal with respect to u^{NE} for each agent $i \in \mathscr{N}$. If, in addition, $\lambda_i \in \Lambda_i \cap \mathbb{R}_{++}^{M_i}$ for all $i \in \mathscr{N}$, then u^{NE} is a Pareto-Nash equilibrium of Γ_M.

5.1 Some Implementation Issues

Assuming common knowledge of Γ_M by all the agents, after exchanging their weighting vectors $\lambda_i,\ i \in \mathscr{N}$, each agent can compute his/her open-loop NE control independently. To overcome the continuous exchange of state information between agents in a closed-loop NE implementation, a receding state-feedback strategy based on the open-loop solution (Gu (2008)) is being developed. Another possibility is the use of local state-observers (Dong (2010)). The problem of selecting proper weighting vectors $\lambda_i,\ i \in \mathscr{N}$, which give rise to the weighting matrices of the FCP, can be formulated in a robust (worst-case) sense (Qu et al. (2015)).

6. ILLUSTRATIVE EXAMPLE

We consider an example discussed in Gu (2008) and Horevicz and Ferreira (2021), which assumes two-dimensional coordinates vectors ($n = 2$) for all the agents and a triangular formation of four agents ($N = 4$), illustrated in Fig. 1.

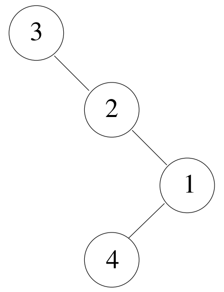

Fig. 1. Graph for the illustrative example.

The triangular formation is characterized by the graph $\mathscr{G} = (\mathscr{N}, \mathscr{E})$ with vertices (agents) $\mathscr{N} = \{1,2,3,4\}$ and edges $\mathscr{E} = \{(1,2),(2,3),(1,4)\}$. The incidence matrix of $\mathscr{G}$ is

$$G = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The weighting matrices (from which all the other matrices are computed) are

$$\begin{aligned} W_{1f} &= W_1 = \mathrm{diag}\,(5,0,5), \\ W_{2f} &= W_2 = \mathrm{diag}\,(5,5,0), \\ W_{3f} &= W_3 = \mathrm{diag}\,(0,5,0), \\ W_{4f} &= W_4 = \mathrm{diag}\,(0,0,5), \\ K_{1f} &= K_1 = \mathrm{diag}\,(1,1,1,1), \\ R_i &= I_2,\ i = 1,2,3,4. \end{aligned}$$

We are interested in analyzing the fundamental conflict between accuracy and energy spent when each agent in a formation tries to keep the desired distances from the his (her) neighbour agents while tracking the reference trajectory. The bi-objective ($M_1 = M_2 = M_3 = M_4 = 2$) cost functionals of the agents are

$$J_{i1}(u) = \frac{1}{2}\|x(T) - r(T)\|_{Q_{if}}^2 + \frac{1}{2}\int_0^T \|x(t) - r(t)\|_{Q_i}^2 dt,$$

$$J_{i2}(u) = \frac{1}{2}\int_0^T \|u_i(t)\|_{R_i}^2 dt, \quad i = 1,2,3,4.$$

The weighted-sum cost functional of each agent can be expressed by a single parameter $\lambda_i \in [0,1)$, in the form

$$J_i^{\lambda_i}(u) = \lambda_i J_{i1}(u) + (1 - \lambda_i) J_{i2}(u), \quad i = 1,2,3,4.$$

The time horizon is $T = 10$ sec and the formation is required to follow a sinusoidal trajectory, that is,

$$r_i = (t, \sin t),\ \rho_i = (0, -\sin t),\ t \geq 0, \quad i = 1,2,3,4.$$

The leader agent, $\ell = 1$, tracks the sinusoid and the other agents keep prescribed distances from their neighbors. Numerical algorithms were implemented in MATLAB, Version R2018b. Nash equilibria were obtained by solving (backwards) the coupled Riccati and auxiliary differential equations and then computing the individual controls according to Theorem 2. The state trajectories presented in Fig. 2 are for the selection $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.5$, that is, all the agents weigh their tracking and energy costs equally. After a transitory period, the desired trajectory was followed in triangular formation with prescribed distances, as required. The associated cost vectors for the formation were

$$J_1(u^{\mathrm{NE}}) = (15.7079, 10.1479, 8.5522, 1.3910)$$

and

$$J_2(u^{\mathrm{NE}}) = (3.6422, 2.3664, 1.5903, 1.9116).$$

In Fig. 3 we present *Pareto-Nash frontiers* of the agents. The frontier of agent i ($i = 1,2,3,4$) was obtained by varying his (her) single weight from 0.1 to 0.9 with increment of 0.1, and keeping constant the weights of the other agents at 0.5. Figs. 4, 5, 6 and 7 show the Pareto-Nash frontiers of the agents in greater detail.

The Pareto-Nash frontiers show that the formation imposes different costs for the agents. The agent 1 must track the reference trajectory and keep prescribed distances from agents 2 and 4. The tracking/formation and energy costs of agent 1 are the greatest. Agents 3 (whose distance to agent 1 is the greatest) and 4 must keep the prescribed distances from agents 2 and 1, respectively. The formation and energy costs of agent 4 are the smallest, while such costs for agent 3 are the second smallest. The costs for the agent 2, which must keep prescribed distances

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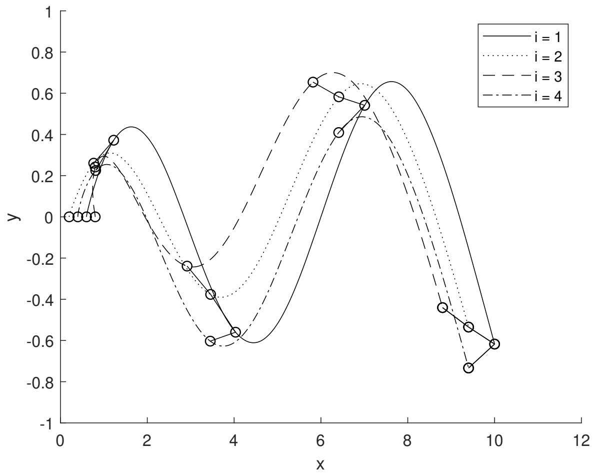

Fig. 2. State trajectories for $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.5$.

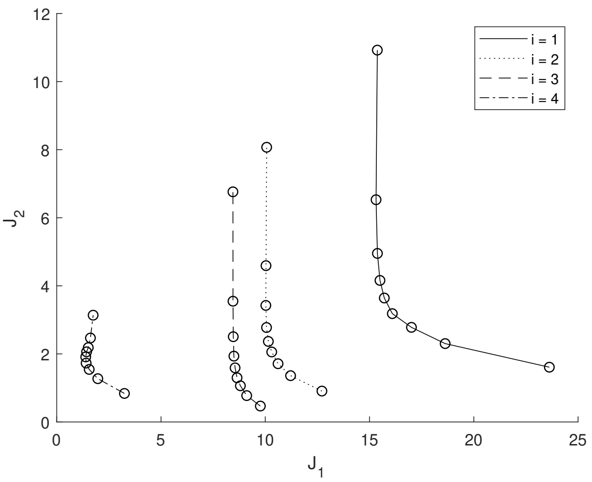

Fig. 3. Pareto-Nash frontiers of the agents 1, 2, 3 and 4.

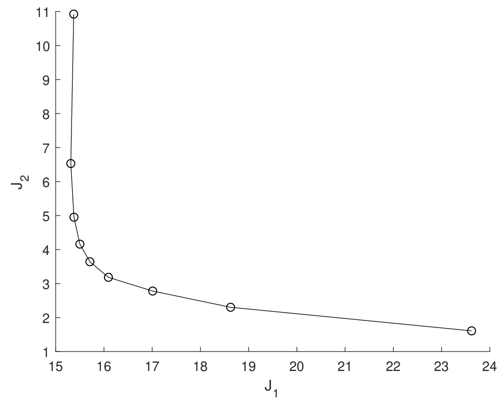

Fig. 4. Pareto-Nash frontier of the leader agent 1.

from agents 1 and 3, are the second greatest. The Pareto-Nash frontiers show (when the weights of the other agents are kept constant at 0.5) that the energy costs of all agents decrease rapidly when their own weights vary from 0.9 to 0.5, approximately, without much effect on the tracking/formation costs. As a second experiment, we selected the alternative weights $\lambda_1 = 0.7$, $\lambda_2 = 0.7$, $\lambda_3 = 0.8$, and $\lambda_4 = 0.5$, aiming at reducing the tracking/formation errors of the agents. The

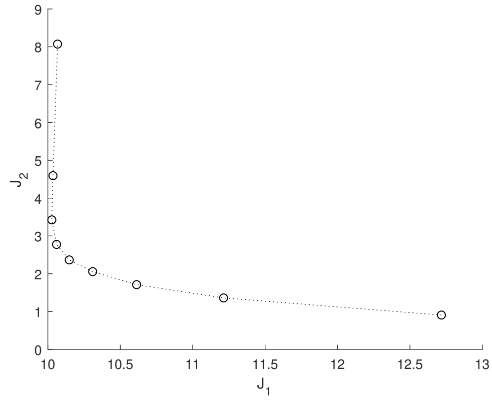

Fig. 5. Pareto-Nash frontier of the agent 2.

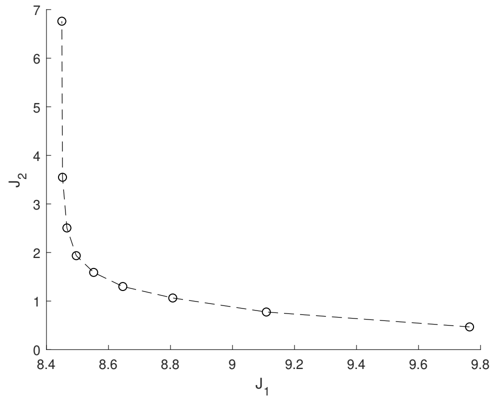

Fig. 6. Pareto-Nash frontier of the agent 3.

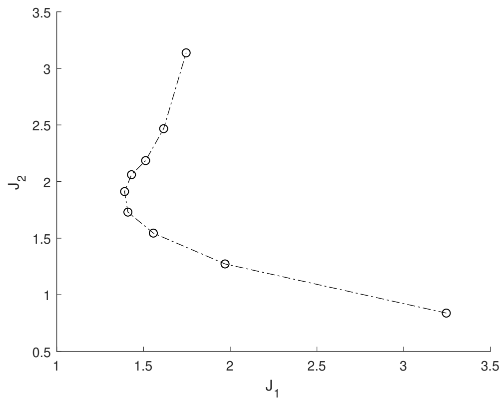

Fig. 7. Pareto-Nash frontier of the agent 4.

corresponding state trajectories are shown in Fig. 8 and the cost vectors for the team of agents were

$$J_1(u^{\text{PN}}) = (15.0695, 9.6682, 8.1779, 1.2747)$$

and

$$J_2(u^{\text{PN}}) = (5.3218, 3.6921, 3.5226, 2.2038).$$

As expected, the cost vector $J_1(u^{\text{PN}})$ dominates $J_1(u^{\text{NE}})$ (in the sense that $J_1(u^{\text{PN}}) < J_1(u^{\text{NE}})$), that is, the alternative control u^{PN} provides smaller tracking/formation errors than u^{NE} for all the agents. That was possible (also as expected) by the increase of energy costs ($J_2(u^{\text{NE}})$ dominates $J_2(u^{\text{PN}})$). The better performance of u^{PN} relative to u^{NE} in terms of tracking errors is more evident (graphically) when the state trajectories of Figs. 2 and 8 are compared in the first 4 sec.

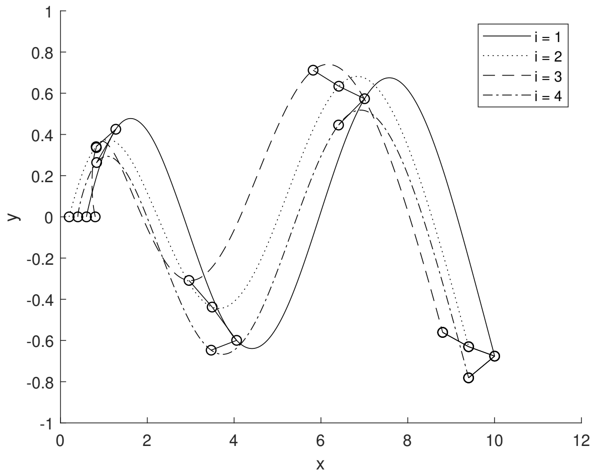

Fig. 8. State trajectories for $\lambda_1 = 0.7$, $\lambda_2 = 0.7$, $\lambda_3 = 0.8$ and $\lambda_4 = 0.5$.

The Nash equilibrium obtained is self-enforcing and Pareto-optimal for *each* agent, that is, with respect to the Nash equilibrium, no other individual control can reduce tracking/formation error without increasing energy cost, and vice-versa. The Pareto-Nash frontiers of the FCP evidence the role of each agent in the formation.

Comparatively to the single objective per agent formulation, the multiobjective formulation proposed provides flexibility for defining objectives and, from the Pareto-Nash frontiers, useful trade-offs (as energy versus cost), which can be used for establishing an informed weighting of the objectives.

7. CONCLUSION

Formation control has been a rich area for the integration of different disciplines as control theory, graph theory and game theory. In this paper we *i)* introduced a multiobjective game-theoretic approach for the problem, *ii)* characterized the Pareto-Nash equilibrium of the resulting multiobjective game, and *iii)* validated both the approach and the solution concept proposed numerically. We believe that the multiobjective formulation represents a significant addition to the literature dedicated to the formation control problem. The authors currently investigate the problem of selecting cost-guaranteed weighting vectors (λ_i, $i \in \mathscr{N}$) for the agents, which would enable a Pareto-Nash robust solution for the formation control problem.

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