Feedback Stabilization of Stochastic Dynamical Systems Using Stochastic Dissipativity Theory

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Abstract: In this paper, the feedback stabilization problem of nonlinear stochastic systems driven by Wiener processes is addressed. It is shown that the existence of a stochastic control Lyapunov function that guarantees the exponential mean square stabilization of the zero solution by linear static output feedback (SOF), under certain circumstances, is equivalent to the stochastic exponential dissipativity of the plant. Quadratic supply rates are proved to be general enough to establish this equivalence. Necessary and sufficient dissipativity-based conditions for stochastic asymptotic stabilization in probability via full state feedback are also given. In general, this work extends recently published results on dissipativity-based feedback stabilization of deterministic nonlinear systems to the problem of stochastic stability analysis and control of stochastic dissipative systems, a research topic within control theory that has been attracting growing interest. An illustrative example is offered as an application of the ideas presented in the article.

Resumo: Neste artigo, é abordado o problema da estabilização em probabilidade de sistemas estocásticos não lineares submetidos a processos de Wiener. Mostra-se que a existência de uma função de Lyapunov estocástica que garanta a estabilização exponencial estocástica no sentido quadrático médio em torno da origem por meio de uma realimentação estática de saída linear é equivalente, sob certas condições, à dissipatividade exponencial estocástica da planta. Prova-se que funções potência quadráticas são suficientemente gerais para que se possa estabelecer tal equivalência. Além disso, condições necessárias e suficientes para a estabilização assintótica em probabilidade via realimentação de estados também são apresentadas. Em geral, este trabalho estende resultados publicados recentemente na literatura sobre a estabilização baseada em dissipatividade de sistemas não lineares determinísticos para o problema de controle de sistemas dissipativos estocásticos, um tópico de pesquisa na área de teoria de controle que tem despertado grande interesse. Um exemplo ilustrativo é oferecido ao final do artigo como uma aplicação das ideias apresentadas no texto.

Keywords: Stochastic Nonlinear Systems, Stabilization in Probability, Stochastic Dissipativity.

Palavras-chave: Sistemas Estocásticos Não Lineares, Estabilização em Probabilidade, Dissipatividade Estocástica.

1. INTRODUCTION

In a state-space framework, dissipativity is described as an input-state-output property based on the notions of generalized storage functions and supply rates (Willems (1972-a), Willems (1972-b)). It has been applied for feedback asymptotic stabilization and exponential stabilization of dynamical systems since its introduction in the field of control theory (Hill and Moylan (1976)). In fact, a storage function, under certain conditions, can be used as a Lyapunov function that guarantees the stability of an equilibrium (Haddad and Chellaboina (2008)). In this regard, it has been recently proved that the exponential stabilization problem of deterministic nonlinear systems around the origin via linear static output feedback (SOF) is equivalent to the system's exponential dissipativity with a quadratic supply rate, subject to a certain equality constraint (see Madeira (2022)). In the same vein, the notion of strict dissipativity was proved necessary and sufficient for closed-loop asymptotic stabilization by full state feedback. By considering continuously differentiable input-affine systems, the work in Madeira (2022) has successfully related converse Lyapunov theorems to the concept of dissipativity of a system. Those results, among many others, provide a strong justification for applying dissipativity theory for feedback stabilization (Brogliato et al (2020), Madeira and Viana (2020), Viana et al (2022), Alves Lima et al (2022), Madeira and Alves Lima (2022)).

Stochastic dissipativity is also defined in literature, and several approaches have been reported on the topic of dissipation inequalities for stochastic dynamical systems (Wu et al (2011), Zhang and Chen (2006), Wu et al (2016) and Wu et al (2012)). For instance, in Rajpurohit and Haddad (2017) a new framework for stochastic dissipativity analysis of nonlinear stochastic systems was introduced, where not only open-loop stability in probability was investigated, but also the stability properties of interconnections of stochastic dissipative systems. Rajpurohit and Haddad (2017) introduced new tools in the field of dissipativity theory, as for example (algebraic) Kalman-Yakubovich-Popov (KYP) conditions for stochastic dissipativity with positive definite energy functions. Sufficient conditions for static and dynamic feedback stabilization in probability of stochastic port-controlled Hamiltonian systems were rather given in Haddad et al (2018). A universal formula for finite-time feedback stabilization of dynamical systems driven by Wiener processes was provided in Haddad and Xin (2020). Classical results on the problem of determining stochastic control Lyapunov functions for stochastic dynamical systems are available in Florchinger (1995), Florchinger (1997) and Chabour and Oumoun (1999). Converse Lyapunov theorems regarding exponential mean square stability in probability are given in Florchinger (1997).

Lately, probabilistic stabilization of networked systems for optimal control through static state feedback was tackled in Bakshi et al (2019) using mean field games. State feedback was also addressed in Bakshi et al (2021) and Bakshi et al (2020), where explicit stability constraints were proposed which are similar to the ones introduced in the present paper. In those publications, however, static output feedback was not investigated and stochastic dissipativity theory was not applied.

In the present article, the dissipativity-based conditions introduced in Madeira (2022) are extended to the case of stochastic dynamical systems using the stochastic dissipativity formulation from Rajpurohit and Haddad (2017). By assuming that stochastic dissipativity is obtained with a quadratic supply rate and a positive definite storage function that is twice continuously differentiable, necessary and sufficient conditions for linear SOF exponential mean square stabilization in probability are proposed for stochastic systems that are also twice continuously differentiable. In addition, a sufficient condition for asymptotic stabilization via linear SOF and new necessary and sufficient conditions for stochastic asymptotic stabilization via state feedback are also provided. In this work, the stabilization of the zero solution $x(t) \overset{a.s.}{\equiv} 0$ is considered.

In Section 2, the notation used in the paper is introduced and, in Section 3, basic statements on stochastic stability and stochastic dissipativity are discussed. In Section 4, the exponential mean square stabilization problem of nonlinear stochastic systems is treated. Section 5 addresses asymptotic stabilization in probability. Section 6 contains an illustrative example, and Section 7 concludes this paper.

2. NOTATION AND MATHEMATICAL BACKGROUND

$\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$ are, respectively, the set of scalars $\beta \in \mathbb{R}$ such that $\beta \geq 0$ and $\beta > 0$. $\mathbb{R}^n$ stands for the set of real column vectors and $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices. Moreover, $(\cdot)^\top$ is the transpose operator, $(\cdot)^{-1}$ is the inverse operator, and $tr(\cdot)$ is the trace operator. x_i is the ith element of a vector $x \in \mathbb{R}^n$. $\mathbb{S}^n$ is the set of real symmetric $n \times n$ matrices. $A > 0$ ($A \geq 0$) means that A is Hermitian positive definite (semidefinite), and $A < 0$ ($A \leq 0$) means that A is negative definite (semidefinite). $||\cdot||$ is the Euclidean norm of a vector or an induced matrix norm (depending on context). $||\cdot||_F$ stands for the Frobenius matrix norm. $f : \mathcal{X} \to \mathcal{Y}$ refers to a function f, a domain $\mathcal{X}$ and a codomain $\mathcal{Y}$. $\mathcal{X} \times \mathcal{Y}$ is the Cartesian product of sets $\mathcal{X}$ and $\mathcal{Y}$. $f : \mathcal{X} \to \mathbb{R}$, $f(0) = 0$, is positive definite (semidefinite) if $f(x) > 0$ ($f(x) \geq 0$) for any $x \in \mathcal{X}$, $x \neq 0$. $f \in C^k$ defines a (vector) function that is k times continuously differentiable with respect to x. The open ball $\mathcal{B}_\delta(x_e)$ is defined as the set $\{x \in \mathbb{R}^n : ||x - x_e|| < \delta\}$, $\delta \in \mathbb{R}_{>0}$.

As a first step, consider a complete probability space $(\mathcal{W}, \mathcal{F}, \mathbb{P})$, where $\mathcal{W}$ is the sample space, $\mathcal{F}$ denotes a σ-algebra, and $\mathbb{P}$ is a probability measure on $\mathcal{F}$, i.e., $\mathbb{P}$ stands for a nonnegative countably additive set function on $\mathcal{F}$ subject to $\mathbb{P}(\mathcal{W}) = 1$ (Arnold (1974)). In addition, let $w(\cdot)$ be a standard d-dimensional *Wiener process* defined by $(w(\cdot), \mathcal{W}, \mathcal{F}, \mathbb{P}^{w_0})$. By considering $\mathbb{P}^{w_0}$ as the classical Wiener measure (Øksendal (1995)), the process $w(t)$ generates a continuous-time filtration $\{\mathcal{F}_t\}_{t\geq 0}$ defined up to time instant t. These are common assumptions in the field of stochastic differential equations and stochastic control (Haddad and Xin (2020)), and will be recurrently referred to in this work.

Then, consider a *stochastic dynamical system* which generates a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ adapted to the stochastic process $x : \mathbb{R}_{\geq 0} \times \mathcal{W} \to \mathcal{X}$ on the space $(\mathcal{W}, \mathcal{F}, \mathbb{P}^{x_0})$, where $\mathcal{X} \subseteq \mathbb{R}^n$ is an open set. Suppose that $\mathcal{F}_\tau \subset \mathcal{F}_t$, $0 \leq \tau \leq t$, in such a manner that $\{\mathcal{W} \in \mathcal{W} : x(t, \mathcal{W}) \in \mathcal{B}\} \in \mathcal{F}_t$, $t \geq 0$, for all possible Borel sets $\mathcal{B} \subset \mathbb{R}^n$ contained in the related Borel σ-algebra $\mathcal{B}^n$. From this point on, the simpler notation $x(t)$ is used to represent the stochastic process $x(t, \mathcal{W})$, as in Haddad et al (2018).

Lastly, let us write $\nabla V(x)^\top \triangleq \partial V(x)/\partial x$ for the Fréchet derivative of $V(x)$ at some point x, and $\nabla^2 V(x) \triangleq \partial^2 V(x)/\partial x^2$ for the Hessian of $V(x)$ at x. Notice that the notation used in this paper for the Fréchet derivative and for the Hessian of $V(x)$ is different from the one used in Haddad and Xin (2020) and in Haddad et al (2018). $\nabla V(x)^\top$ and $\nabla^2 V(x)$ are applied here in order to be compatible with we notation employed in Madeira (2022). In addition, $\mathcal{H}_n$ denotes the Hilbert space of random vectors $x \in \mathbb{R}^n$ with some finite average power, or equivalently, $\mathcal{H}_n \triangleq \left\{x : \mathcal{W} \to \mathbb{R}^n; \mathbb{E}[x^\top x] < \infty\right\}$, where $\mathbb{E}$ stands for the expectation of a random variable. For an open set $\mathcal{X} \subseteq \mathbb{R}^n$, another important definition is given by the set $\mathcal{H}_n^{\mathcal{X}} \triangleq \{x \in \mathcal{H}_n : x : \mathcal{W} \to \mathcal{X}\}$, which represents all the random vectors in $\mathcal{H}_n$ induced by $\mathcal{X}$. Furthermore, for every $x_0 \in \mathbb{R}^n$, $\mathcal{H}_n^{x_0} \triangleq \left\{x \in \mathcal{H}_n : x \overset{a.s.}{=} x_0\right\}$ (Haddad et al (2018)).

3. PRELIMINARIES AND DEFINITIONS

3.1 Stochastic Systems and Stochastic Stability

The following controlled nonlinear stochastic system (Haddad and Xin (2020)) is considered in this section

$$\begin{aligned} dx(t) &= F(x(t), u(t))dt + D(x(t))dw(t) \\ &= [f(x(t)) + g(x(t))u(t)]dt + D(x(t))dw(t), \end{aligned} \quad (1)$$

$x(t_0) \overset{a.s.}{=} x_0$, $t \geq t_0$. For every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{X}}$ is an $\mathcal{F}_t$-measurable random state vector, with $x(t_0) \in \mathcal{H}_n^{x_0}$. $\mathcal{X} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{X}$ and, as stated in the previous section, $w(t)$ is a d-dimensional independent standard Wiener process defined on a complete filtered probability space $(\mathcal{W}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P})$. In this work, $f : \mathcal{X} \to \mathbb{R}^n$, $g : \mathcal{X} \to \mathbb{R}^{n \times m}$ and $D : \mathcal{X} \to \mathbb{R}^{n \times d}$ are continuous functions. Besides, $f(0) = 0$ and $D(0) = 0$ imply that $(x(t), u(t)) \overset{a.s.}{\equiv} (0, 0)$ is an equilibrium solution of (1) (Rajpurohit and Haddad (2017)). Furthermore, assume that the input $u(\cdot)$ takes values in a compact and metrizable set $\mathcal{U} \subseteq \mathbb{R}^m$ and, again, that $F(\cdot, \cdot)$ is continuous in a neighborhood $\mathcal{X} \times \mathcal{U}$ of the origin.

Suppose that $u(t) \in \mathcal{H}_m^{\mathcal{U}}$, $\mathcal{U} \subseteq \mathbb{R}^m$, verifies certain regularity conditions which guarantee the existence of a pathwise unique solution for (1) in $(\mathcal{W}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P}^{x_0})$. That is, assume that $u(\cdot)$ belongs to the class of admissible controls consisting in measurable functions $u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t \geq t_0}$. Furthermore, suppose that for all $t \geq s$, $(w(t) - w(s))$ is independent of $u(\tau)$, $w(\tau)$, $\tau \leq s$, and $x(t_0)$, that is, the control $u(\cdot)$ is nonanticipative (Haddad et al (2018)). In addition, for every $t \geq t_0$, an output signal is given by

$$y(t) = h(x(t)), \quad (2)$$

where $y(t) \in \mathcal{H}_l^{\mathcal{Y}}$, $\mathcal{Y} \subseteq \mathbb{R}^l$, and $h : \mathcal{X} \to \mathbb{R}^l$, $h(0) = 0$, is a function that can be used for feedback stabilization.

Besides, consider that the following conditions hold uniformly in $u \in \mathcal{U}$, for all $(x, y) \in \mathcal{X}$ and some Lipschitz constant $L \in \mathbb{R}_{>0}$ (Haddad et al (2018))

$$||F(x, \cdot) - F(y, \cdot)|| + ||D(x) - D(y)||_F \leq L||x - y|| \quad (3)$$

$$||F(x, \cdot)||^2 + ||D(x)||_F^2 \leq L^2(1 + ||x||^2). \quad (4)$$

Then, from Khasminskii (2012) the unique solution $x(t) \in \mathcal{H}_n^{\mathcal{X}}$ of (1) is a time-homogeneous Feller continuous Markov process with a stationary Feller transition probability function.

Definition 1. (see Øksendal (1995)) Consider a time-homogeneous Markov process $x(t) \in \mathcal{H}_n^{\mathcal{X}}$ and a function $V : \mathcal{X} \to \mathbb{R}$. The *infinitesimal generator* $\mathcal{L}$ of $x(t)$, $t \geq 0$, with $x(0) \overset{a.s.}{=} x_0$, is defined by

$$\mathcal{L}V(x_0) \triangleq \underset{t \to 0^+}{lim} \frac{\mathbb{E}^{x_0}[V(x(t))] - V(x_0)}{t}, \quad x_0 \in \mathcal{X}, \quad (5)$$

where $\mathbb{E}^{x_0}$ denotes the expectation with respect to the probability measure $\mathbb{P}^{x_0}(x(t) \in \mathcal{B}) \triangleq \mathbb{P}(t_0, x_0, t, \mathcal{B})$.

According to Rajpurohit and Haddad (2017), if $V \in C^2$ and has a compact support, and if the process $x(t)$ satisfies (1) for all $t \geq 0$, then for all $x \in \mathcal{X}$ the infinitesimal generator $\mathcal{L}$ of $x(t)$ is given by

$$\mathcal{L}V(x) \triangleq \nabla V(x)^\top F(x, u) + \frac{1}{2} tr[D^\top(x) \nabla^2 V(x) D(x)]. \quad (6)$$

Then, to tackle the feedback stabilization problem, let us call a mapping $\eta : \mathcal{X} \to \mathcal{U}$ a *control law*, where η is measurable and subject to $\eta(0) = 0$. If under $u(t) = u(x(t)) = \eta(x(t))$, $t \geq t_0$, the process $x(t)$ fulfills (1), then $u(\cdot)$ is said to be a *feedback control law* (Haddad and Xin (2020)). Thus, as $u(\cdot)$ is an admissible control, due to the fact that $\eta(\cdot)$ has values in $\mathcal{U}$, the closed-loop system of (1) is well-defined

$$\begin{aligned} dx(t) &= [f(x(t)) + g(x(t))\eta(x(t))]dt + D(x(t))dw(t) \\ &= F(x(t)) + D(x(t))dw(t), \end{aligned} \quad (7)$$

$x(t_0) \overset{a.s.}{=} x_0$, $t \geq t_0$, where with $u(x(t)) = \eta(x(t))$,

$$F(x(t)) = f(x(t)) + g(x(t))u(x(t)). \quad (8)$$

In this case, conditions (3)-(4) are functions of the state alone and hold for $x \in \mathcal{X}$, which implies the existence and uniqueness of solutions of nonlinear equations (7).

Definition 2. (see Khasminskii (2012)) *i)* The zero solution $x(t) \overset{a.s.}{\equiv} 0$ to (7) is *Lyapunov stable in probability* if, for every $(\varepsilon, \rho) \in \mathbb{R}_{>0}$, there exist $\delta = \delta(\rho, \varepsilon) \in \mathbb{R}_{>0}$ such that, for all $x_0 \in \mathcal{B}_\delta(0)$,

$$\mathbb{P}^{x_0}\left(sup_{t \geq t_0}||x(t)|| > \varepsilon\right) \leq \rho. \quad (9)$$

ii) The zero solution $x(t) \overset{a.s.}{\equiv} 0$ to (7) is *asymptotically stable in probability* if it is Lyapunov stable in probability and there exists $\delta \in \mathbb{R}_{>0}$ such that if $x_0 \in \mathcal{B}_\delta(0)$, then

$$\mathbb{P}^{x_0}\left(\underset{t \to +\infty}{lim}||x(t)|| = 0\right) = 1. \quad (10)$$

If, in addition, (10) holds for all $x_0 \in \mathbb{R}^n$, then the zero solution $x(t) \overset{a.s.}{\equiv} 0$ to (1) is *globally asymptotically stable in probability.*

iii) The zero solution $x(t) \overset{a.s.}{\equiv} 0$ to (7) is *exponentially mean square stable in probability* if there exist scalars $(\alpha, \beta, \delta) \in \mathbb{R}_{>0}$, such that if $x_0 \in \mathcal{B}_\delta(0)$, then

$$\mathbb{E}^{x_0}[||x(t)||^2] \leq \alpha||x_0||^2 e^{-\beta t}. \quad (11)$$

If, in addition, (11) holds for all $x_0 \in \mathbb{R}^n$, then the zero solution $x(t) \overset{a.s.}{\equiv} 0$ to (7) is *globally exponentially mean square stable in probability.*

Notice that the stability properties introduced in Definition 2 hold in some open ball $x_0 \in \mathcal{B}_\delta(0)$, where $\mathcal{B}_\delta(0) \subset \mathcal{X}$. Conversely, from Haddad and Xin (2020), a given nonlinear stochastic system (1) under (3)-(4) can be held stochastically asymptotically stable by a feedback $u(x(t)) = \eta(x(t))$ around the zero solution $x(t) \overset{a.s.}{\equiv} 0$ if and only if there exists a *stochastic control Lyapunov function*, $V > 0$, $V \in C^2$, satisfying

$$\nabla V(x)^\top F(x) + \frac{1}{2} tr[D^\top(x) \nabla^2 V(x) D(x)] < 0, \quad (12)$$

for all $x \in \mathcal{B}_\delta(0) \subset \mathcal{X}$, $x \neq 0$.

Remark 1. In the remainder of this paper, asymptotic feedback stabilization means the fulfillment of condition (12), where $F(x)$ is defined in (8). This condition only demands that $(f, g, D) \in C^0$, i.e., that they are continuous.

In literature, $u(x)$ is typically called a state feedback in the case where all state components can be measured and are readily available for feedback. This represents a best case scenario for control and, quite frequently, is not the situation one encounters in practice. Thus, in real-world applications, output feedback control might be the case, where in this context a static output feedback (SOF) provides the simplest possible approach for controller design. For instance, the problem of linear SOF stabilization of stochastic nonlinear systems can be solved through the same tools one applies for state feedback, with the difference that in this case the control signal is constrained to a certain structure $u(x) = u(h(x)) = Kh(x)$, where $K \in \mathbb{R}^{m\times l}$ is a constant and stabilizing gain one needs to determine.

Lastly, assume that the equilibrium $x(t) \overset{a.s.}{\equiv} 0$ of the stochastic differential equation (7) is held exponentially mean square stable in probability by some feedback $u(x(t)) = \eta(x(t))$ in $x \in \mathcal{B}_\delta(0) \subset \mathcal{X}$. Suppose, in this case, that $(f, g, D) \in C^2$. Then, there exist a stochastic control Lyapunov function $V \in C^2$ and $\alpha_i \in \mathbb{R}_{>0}$, $1 \le i \le 5$, such that for all $x \in \mathcal{B}_\delta(0)$ (Florchinger (1997))

$$\nabla V(x)^\top F(x) + \frac{1}{2}tr[D^\top(x)\nabla^2 V(x)D(x)] \le -\alpha_1 V(x), \tag{13}$$

and

$$\alpha_2||x||^2 \le V(x) \le \alpha_3||x||^2, \tag{14}$$

$$||\nabla V(x)|| \le \alpha_4||x||, \tag{15}$$

$$||\nabla^2 V(x)||_F \le \alpha_5. \tag{16}$$

Remark 2. As in this work feedback stabilization stands out as the major aim of applying stochastic dissipativity theory, the problem of determining a control signal that exponentially stabilizes (7) is what is meant when conditions (13)-(16) are called upon, subject to $V > 0$, $V \in C^2$. In other words, fulfilling those conditions should be understood in the remainder of this paper as a synonym for the exponential feedback stabilization problem of (7).

3.2 Stochastic Dissipativity Theory

For a stochastic system (1)-(2), a real function $r : \mathcal{U}\times\mathcal{Y} \to \mathbb{R}$, with $r(0, 0) = 0$, is called a supply rate if for all $u(\cdot) \in \mathcal{U}$ and $y(\cdot) \in \mathcal{Y}$ satisfying (1)-(2), $r(u(t), y(t))$ fulfills the requirement that $\mathbb{E}\Big[\int_{t_1}^{t_2} |r(u(s), y(s))|ds\Big] < +\infty$, t_1, $t_2 \ge 0$ (Rajpurohit and Haddad (2017)). A storage function $V : \mathcal{X} \to \mathbb{R}$, with $V \ge 0$, is a function that represents the generalized energy stored inside a system. If one has $V \in C^2$ and the system is completely stochastically reachable (Rajpurohit and Haddad (2017)), then a mathematical formulation for the stochastic dissipativity of (1)-(2) with respect to the supply rate $r(u, y)$ can be characterized by the infinitesimal generator $\mathcal{L}V(x)$ given in (6). Specifically, if V has a compact support, then (1)-(2) is stochastically dissipative with respect to $r(u, y)$ if and only if

$$\nabla V(x)^\top[f(x) + g(x)u] + \frac{1}{2}tr[D^\top(x)\nabla^2 V(x)D(x)] \le r(u, y), \tag{17}$$

for all $(x, u) \in \mathcal{X} \times \mathcal{U}$. In a similar manner, a necessary and sufficient condition for the stochastic exponential dissipativity of (1)-(2) in $(x, u) \in \mathcal{X} \times \mathcal{U}$ is as follows (Rajpurohit and Haddad (2017))

$$\nabla V(x)^\top[f(x) + g(x)u] + \frac{1}{2}tr[D^\top(x)\nabla^2 V(x)D(x)] + \varepsilon V(x) \le r(u, y). \tag{18}$$

Then, by considering a quadratic supply rate given by

$$r(u, y) = y^\top Qy + 2y^\top Su + u^\top Ru, \tag{19}$$

$Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, extended Kalman-Yakubovich-Popov conditions can be derived for stochastic dynamical systems, see Rajpurohit and Haddad (2017, Remark 4.1). As a result, the system (1)-(2) is said to be stochastically dissipative with respect to (19) if there exists $V : \mathcal{X} \to \mathbb{R}$, $V \in C^2$, $V > 0$, such that for all $x \in \mathcal{X}$

$$\nabla V^\top f + \frac{1}{2}tr\Big[D^\top\nabla^2 VD\Big] - h^\top Qh + \left[\frac{1}{2}\nabla V^\top g - h^\top S\right]R^{-1}\left[\frac{1}{2}\nabla V^\top g - h^\top S\right]^\top \le 0. \tag{20}$$

This means that (17) holds for all $(x, u) \in \mathcal{X} \times \mathbb{R}^m$. If relation (20) is replaced by

$$\nabla V^\top f + \frac{1}{2}tr\Big[D^\top\nabla^2 VD\Big] + \varepsilon V - h^\top Qh + \left[\frac{1}{2}\nabla V^\top g - h^\top S\right]R^{-1}\left[\frac{1}{2}\nabla V^\top g - h^\top S\right]^\top \le 0, \tag{21}$$

where $\varepsilon \in \mathbb{R}_{>0}$, then one has a sufficient condition for stochastic exponential dissipativity in $(x, u) \in \mathcal{X} \times \mathbb{R}^m$. Notice that (20) and (21) are functions of the state x alone and do not depend on the control u.

4. EXPONENTIAL MEAN SQUARE STABILIZATION

Following the steps of Madeira (2022), new necessary and sufficient stabilizability conditions for stochastic systems can be derived using dissipativity theory. Exponential stabilization through linear SOF, for example, can be tackled if a few assumptions are made. In the present paper, though, stability is regarded as the stability in probability of what is called as the zero solution $x(t) \overset{a.s.}{\equiv} 0$ of (1). Then, suppose that (1) is held exponentially mean square stable in probability by some linear static output feedback $u(x) = Kh(x)$, where $h(x)$ is given in (2). By employing a converse Lyapunov theorem from stochastic stability theory, it is proved in this section that a stochastic control Lyapunov function for the closed-loop dynamics $[f(x) + g(x)u(x)]dt + D(x)dw$ subject to (13)-(16) *exists* if, and only if, the system representation (1)-(2) subject to (14)-(16) is stochastically exponentially dissipative, and an equality condition on (Q, S, R) holds. The following theorem and its proof are similar to the results found in Madeira (2022), although it deals with the stochastic case. In this section, $(f, g, D, h, V) \in C^2$.

Theorem 1. Consider a stochastic system (1)-(2). This system is exponentially mean square stabilizable by some linear SOF if and only if it is stochastically exponentially dissipative with $\Delta_c = 0$, where

$$\Delta_c = SR^{-1}S^\top - Q, \tag{22}$$

and $R > 0$. A stabilizing SOF is given by

$$u = Ky, \quad K = -R^{-1}S^\top. \tag{23}$$

Proof: Necessity: Once the stochastic system (1)-(2) is made exponentially mean square stable about the zero solution $x(t) \overset{a.s.}{\equiv} 0$ by some linear static output feedback $u(x) = Kh(x)$, it follows that (13)-(16) necessarily hold in an open ball $\mathcal{B}_\delta(0)$, $\delta \in \mathbb{R}_{>0}$, for a certain set of scalars

$\alpha_i \in \mathbb{R}_{>0}$, $1 \leq i \leq 5$, and for some Lyapunov function $V \in C^2$ which is guaranteed to exist.

Due to the fact that Lyapunov condition (13) is known to be feasible, the following inequality holds for all $x \in \mathcal{B}_\delta(0)$, $x \neq 0$, for some $\bar{\varepsilon} \in \mathbb{R}_{>0}$, $0 < \bar{\varepsilon} < \alpha_1$,

$$\nabla V^\top [f + gKh] + \frac{1}{2} tr[D^\top \nabla^2 V D] + \bar{\varepsilon} V < 0. \quad (24)$$

If one considers, for example, $\bar{\varepsilon} = \alpha_1/2$, then it follows from (13) that the subsequent relation holds

$$\nabla V^\top [f + gKh] + \frac{1}{2} tr[D^\top \nabla^2 V D] + \bar{\varepsilon} V \leq -\bar{\varepsilon} V < 0. \quad (25)$$

Since $V \in C^2$, one has that $\nabla V \in C^1$ and, by assumption, relation (15) holds for some α_4. Thus, there exists some small enough $\beta \in \mathbb{R}_{>0}$ which guarantees that for all $x \in \mathcal{B}_\delta(0)$

$$\nabla V^\top [f + gKh] + \frac{1}{2} tr[D^\top \nabla^2 V D] + \bar{\varepsilon} V + \frac{\beta}{4} \nabla V^\top g g^\top \nabla V \leq 0, \quad (26)$$

which is equivalent to,

$$\nabla V^\top f + \frac{1}{2} tr[D^\top \nabla^2 V D] + \bar{\varepsilon} V \leq -\nabla V^\top gKh - \frac{\beta}{4} \nabla V^\top g g^\top \nabla V. \quad (27)$$

Notice that from (25) and (26) one obtains

$$\nabla V^\top [f + gKh] + \frac{1}{2} tr[D^\top \nabla^2 V D] + \bar{\varepsilon} V + \frac{\beta}{4} \nabla V^\top g g^\top \nabla V \leq -\bar{\varepsilon} V + \frac{\beta}{4} \nabla V^\top g g^\top \nabla V \quad (28)$$

and from equations (18)-(21) of Madeira (2022, Theorem 1), one is always able to determine some $\beta \in \mathbb{R}_{>0}$ which leads to $-\bar{\varepsilon} V + (\beta/4) \nabla V^\top g g^\top \nabla V \leq 0$ for all $x \in \mathcal{B}_\delta(0)$. The same arguments used in Madeira (2022) apply here since $(f, g, D, h, V) \in C^2$, whereas in that paper the weaker condition $(f, g, h, V) \in C^1$ was considered.

Then let us connect stability condition (28) with stochastic dissipativity. Suppose that condition (21) holds for all $x \in \mathcal{B}_\delta(0)$, with $V \in C^2$ subject to (14)-(16), and consider $\varepsilon = \bar{\varepsilon}$. This is a sufficient condition for stochastic exponential dissipativity in $(x, u) \in \mathcal{B}_\delta(0) \times \mathbb{R}^m$, and from (21) one has

$$\nabla V^\top f + \frac{1}{2} tr[D^\top \nabla^2 V D] + \bar{\varepsilon} V \leq h^\top Q h - \frac{1}{4} \nabla V^\top g R^{-1} g^\top \nabla V + \frac{1}{2} \nabla V^\top g R^{-1} S^\top h + \frac{1}{2} h^\top S R^{-1} g^\top \nabla V - h^\top S R^{-1} S^\top h. \quad (29)$$

Exponential mean square stabilizability by linear SOF implies the feasibility of stochastic exponential dissipativity condition (21) if the right-hand side of (27) is not greater than the right-hand side of (29), i.e., if

$$-\nabla V^\top gKh - \frac{\beta}{4} \nabla V^\top g g^\top \nabla V \leq h^\top Q h - \frac{1}{4} \nabla V^\top g R^{-1} g^\top \nabla V + \frac{1}{2} \nabla V^\top g R^{-1} S^\top h + \frac{1}{2} h^\top S R^{-1} g^\top \nabla V - h^\top S R^{-1} S^\top h. \quad (30)$$

This is exactly the same condition obtained in Madeira (2022, eq. (24)) (the deterministic case), where this relation is known to be true if we set $R = \frac{1}{\beta} I > 0$, $K = -R^{-1} S^\top$ ($S = -K^\top R$) and $Q = S R^{-1} S^\top$. Consequently, if the stochastic system (1)-(2) is held exponentially mean square stable about the zero solution by some linear static output feedback, then it inevitably fulfills condition (21) with $R > 0$ and $\Delta_c = 0$, which in turn means that the system is stochastically exponentially dissipative in $(x, u) \in \mathcal{B}_\delta(0) \times \mathbb{R}^m$.

Sufficiency: From (18) and (19), (1)-(2) is stochastically exponentially dissipative if

$$\nabla V^\top [f + gu] + \frac{1}{2} tr[D^\top \nabla^2 V D] + \varepsilon V \leq y^\top Q y + 2 y^\top S u + u^\top R u. \quad (31)$$

Taking into account that $y = h$, relation (31) can be considered as a function defined in a domain $(x, u) \in \mathcal{X} \times \mathcal{U}$ around $(x(t), u(t)) \overset{a.s.}{\equiv} (0, 0)$. Suppose, moreover, that $V \in C^2$ is a positive definite function and also verifies (14)-(16) in some subset of $\mathcal{X}$.

With $h(0) = 0$, $\Delta_c \geq 0$ and a control signal given by (23), condition (31) yields

$$\nabla V^\top [f + gKh] + \frac{1}{2} tr[D^\top \nabla^2 V D] + \varepsilon V \leq -h^\top \Delta_c h, \quad (32)$$

for all $x \in \mathcal{B}_\delta(0)$, where $\mathcal{B}_\delta(0) \subset \mathcal{X}$ is the largest open ball entirely inside the closed-loop domain of attraction such that $x \in \mathcal{B}_\delta(0)$ implies $u(x) \in \mathcal{U}$. This means that stochastic dissipativity with $\Delta_c = 0$ is also a sufficient condition for feedback stabilization in probability, as (32) implies Lyapunov condition (13). Thus, the zero solution $x(t) \overset{a.s.}{\equiv} 0$ of (1)-(2) is exponentially mean square stabilizable by linear static output feedback if and only if the system is stochastically exponentially dissipative subject to $\Delta_c = 0$ and $R > 0$. □

5. ASYMPTOTIC STABILIZATION IN PROBABILITY

The results of Section III can be adapted to derive a sufficient condition for obtaining asymptotic stabilization in probability using SOF. Such condition, unfortunately, could not be proved necessary. Suppose that $V > 0$, $V \in C^2$, and consider the following stochastic dissipativity condition in $(x, u) \in \mathcal{X} \times \mathcal{U}$,

$$\nabla V^\top [f + gu] + \frac{1}{2} tr[D^\top \nabla^2 V D] + T \leq h^\top Q h + 2 h^\top S u + u^\top R u, \quad (33)$$

with some $T(x) > 0$, $x \neq 0$, $T(0) = 0$. If $T(x) = \varepsilon V(x)$, $\varepsilon \in \mathbb{R}_{>0}$, then (18) is recovered. In this section, $(f, g, D, h) \in C^0$ subject to (3)-(4).

Corollary 1. Suppose that $R > 0$ and Δ_c is given by (22). If system (1)-(2) fulfills (33) with $\Delta_c \geq 0$, then the zero solution is asymptotically stabilizable in probability by the linear SOF (23).

Proof: Under (22) and (23), condition (33) results in

$$\nabla V^\top [f + gKh] + \frac{1}{2} tr[D^\top \nabla^2 V D] \leq -T - h^\top \Delta_c h. \quad (34)$$

Due to (12), if $\Delta_c \geq 0$ and $h(0) = 0$, then the zero solution is asymptotically stable in probability for all $x(0) \in \mathcal{B}_\delta(0)$, where $\mathcal{B}_\delta(0) \subset \mathcal{X}$ is the largest open ball entirely inside the closed-loop domain of attraction such that $x \in \mathcal{B}_\delta(0)$ implies $u(x) \in \mathcal{U}$. □

Similarly to what was done in Madeira (2022), the problem of designing a state feedback for asymptotic stabilization of system (1) is equivalent to the task of specifying an appropriate and possibly fictitious output variable (2) which renders the state-space representation (1)-(2) linear SOF stabilizable in accordance with the conditions of Corollary 1. With regards to state feedback, though, stochastic dissipativity is proved to be both a necessary and sufficient condition for asymptotic stabilization in probability. Firstly, consider the following strict relation valid for all $x \in \mathcal{X}$, with $R > 0$, which is sufficient for (33) in $(x, u) \in \mathcal{X} \times \mathbb{R}^m$,

$$\nabla V^\top f + \frac{1}{2} tr(D^\top \nabla^2 V D) + T - h^\top Q h + \left[\frac{1}{2}\nabla V^\top g - h^\top S\right] R^{-1} \left[\frac{1}{2}\nabla V^\top g - h^\top S\right]^\top \le 0. \quad (35)$$

Theorem 2. Consider a stochastic system given by (1). This system is asymptotically stabilizable in probability through some state feedback if and only if there exists an (fictitious) output variable (2) such that (33) holds with $\Delta_c = 0$, where

$$\Delta_c = SR^{-1}S^\top - Q, \quad (36)$$

and $R > 0$. A stabilizing state feedback is given by

$$u(x) = -R^{-1}S^\top h(x). \quad (37)$$

Proof: Necessity: The proof is a simple extension of the results of Madeira (2022, Theorem 3) to the stochastic case. Firstly, owing to the fact that system (1) is, by assumption, asymptotically stabilizable in probability by some state feedback $u(x)$, it follows that Lyapunov condition (12) is feasible in some $\mathcal{B}_\delta(0) \subset \mathcal{X}$, or equivalently, there exists some $V \in C^2$, $V > 0$, $T > 0$, such that for all $x \in \mathcal{B}_\delta(0)$, $x \neq 0$,

$$\nabla V(f + gu) + \frac{1}{2} tr[D^\top \nabla^2 V D] + T \le 0. \quad (38)$$

From the same reasoning applied in Madeira (2022, Theorem 3), one concludes that for all $\beta \in \mathbb{R}_{>0}$ the following inequality holds in the previously mentioned open ball $\mathcal{B}_\delta(0)$,

$$\nabla V^\top [f + g\overline{u}] + \frac{1}{2} tr[D^\top \nabla^2 V D] + T + \frac{\beta}{4} \nabla V^\top g g^\top \nabla V \le 0, \quad (39)$$

$$\overline{u}(x) = u(x) - \frac{\beta}{4} g(x)^\top \nabla V(x). \quad (40)$$

Next, define a suitable output variable $h(x) = \overline{u}(x)$, with $\overline{u} = Kh$, $K = I$. Without loss of generality $R = \frac{1}{\beta} I > 0$, and (37) holds if $-R^{-1}S^\top = I \Rightarrow S = -\frac{1}{\beta} I$.

Furthermore, dissipativity condition (35) is equivalent to

$$\nabla V^\top f + \frac{1}{2} tr[D^\top \nabla^2 V D] + T \le h^\top Q h - \frac{1}{4} \nabla V^\top g R^{-1} g^\top \nabla V + \frac{1}{2} \nabla V^\top g R^{-1} S^\top h + \frac{1}{2} h^\top S R^{-1} g^\top \nabla V - h^\top S R^{-1} S^\top h. \quad (41)$$

Finally, stochastic stability implies stochastic dissipativity if condition (30) holds, which is true if $K = -R^{-1}S^\top$ and $\Delta_c = 0$, that is $Q = SR^{-1}S^\top$. Thus, stabilizability by full state feedback implies the feasibility of (33) with $R > 0$ and $\Delta_c = 0$, where an output is given by $h(x) = \overline{u}(x)$.

Sufficiency: It can be readily derived from Corollary 1. Given that every state variable is directly measured and are deployed for feedback, then the state-dependent function $u(x)$ is defined as a state feedback control law. □

If (f, g, h, D) are not in C^2, then the conditions presented in Theorem 2 are only sufficient for stabilization.

6. AN ILLUSTRATIVE EXAMPLE

The controlled stochastic Duffing equation (Rajpurohit and Haddad (2017)) is considered, where

$$f(x) = \begin{bmatrix} x_2 \\ -2x_1 - x_1^3 - cx_2 \end{bmatrix}, \; g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; D(x) = \begin{bmatrix} 0 \\ \sigma x_2 \end{bmatrix}.$$

Then, suppose that a stabilizing linear SOF has to be designed, with $h(x) = x_2$.

Dissipativity condition (35) holds locally with $R \in \mathbb{R}_{>0}$ if

$$\frac{1}{2} \nabla V^\top g = h^\top S, \quad (42)$$

$$\nabla V^\top f + \frac{1}{2} tr(D^\top \nabla^2 V D) + T - h^\top Q h \le 0. \quad (43)$$

As $\nabla V = [V_{x_1} \; V_{x_2}]^\top$, (42) leads to $\frac{1}{2} V_{x_2} = Sx_2$, which means that $S \in \mathbb{R}_{>0}$ and $V(x) = V_1(x_1) + Sx_2^2$, for some $V_1(x_1) > 0$. Thus, condition (43) is equivalent to

$$x_2 V_{x_1} - 4Sx_1x_2 - 2Sx_1^3x_2 - 2Scx_2^2 + S\sigma^2 x_2^2 - Qx_2^2 + T. \quad (44)$$

If $V_1(x_1) = 2Sx_1^2 + \frac{1}{2}Sx_1^4$, then (44) is equal to

$$S(\sigma^2 - 2c)x_2^2 - Qx_2^2 + T, \quad (45)$$

that is nonpositive for $T = 0$ and some sufficiently large $Q \in \mathbb{R}_{>0}$. Then, although (43) is not feasible, one obtains $\Delta_c = 0$ and $\mathcal{L}V(x) \le 0$ for some large enough R^{-1}. Moreover, for $x_2 = 0$, if follows that $dx_2 = 0 \Leftrightarrow x_1 = 0$, which from Haddad et al (2018) implies that the linear SOF given by (37), i.e., $u = -R^{-1}Sx_2$, stochastically asymptotically stabilizes the zero solution $x(t) \overset{a.s.}{\equiv} 0$ in probability.

In Rajpurohit and Haddad (2017), $c \ge \frac{1}{2}\sigma^2$ was a requirement for verifying the stochastic passivity of the plant and for designing a dynamic output feedback that achieved the global asymptotic stabilization in probability of the plant-controller interconnection. Here, asymptotic stabilization by linear SOF is possible even if $c < \frac{1}{2}\sigma^2$, as long as appropriate Q and R are considered. This is another main advantage of dissipativity-based control when compared to its passivity-based counterpart. Manipulating the set of matrices (Q, S, R) offers a great degree of flexibility for designing a stabilizing controller. As in the deterministic case, passivity is not necessary for stabilization, whereas QSR-dissipativity seems to be a more appropriate notion for feedback control.

7. CONCLUSION AND FUTURE RESEARCH

It has been proved in this paper that the results introduced in Madeira (2022) can be extended to the case of stochastic dynamical systems driven by Wiener processes. The linear SOF (local) exponential mean square stabilizability of the zero solution is equivalent to the (local) stochastic exponential dissipativity of the input-affine system considered, under certain conditions. By supposing that all storage and Lyapunov functions involved are positive definite C^2 functions and fulfill conditions (14)-(16), the converse Lyapunov results found in Florchinger (1997) were shown to be equivalent to a stochastic dissipativity condition subject to $R > 0$ and $\Delta_c = SR^{-1}S^\top - Q = 0$. The problem of stochastic asymptotic stabilization of nonlinear systems by full state feedback has also been solved in terms of necessary and sufficient dissipativity-based conditions. These new results provide further justification for the application of dissipativity for feedback stabilization.

Future research on this topic are likely to include the feedback stabilization in probability of nonzero solutions of (1), and the use of a possibly nonlinear SOF. Another interesting topic of future research could be the case of the discrete-time nonlinear stochastic systems, as well as a specific sum of squares (SOS) strategy for polynomial systems. Optimality, in addition, should also be addressed in upcoming publications.

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