Switched Control via Static Output Feedback for Uncertain LTI Systems

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Abstract: In this paper, a new switched static output feedback controller design for uncertain linear time-invariant (LTI) systems is proposed. The approach chosen for the design of static output feedback gains is based on the two-stage method, which consists in first obtaining a state feedback gain matrix that is then used as an input parameter for the design of the desired static output feedback controller at the second stage. The proposed strategy considers performance improvement regarding the specification of a minimum closed-loop rate. The solution for the investigated problem is presented in terms of linear matrix inequalities (LMIs) obtained using the Finsler's Lemma. The obtained results are compared with a robust static output feedback method, in a controller design and practical implementation for a real active suspension system.

Keywords: Static output feedback control; Switched control; Uncertain linear time-invariant systems; Robust control; Linear matrix inequalities (LMIs).

1. INTRODUCTION

The state feedback control technique is not possible to be directly implemented in practice when all states are not available for measurement, which makes the design of an output feedback controller important to deal with this kind of situation (Kimura, 1975). Essentially, there are two methods for development and control with output feedback, the first one is called dynamic output feedback (DOF), which consists in developing a feedback loop with own dynamics (Zhai and Liu, 2021). The second method, referred to as static output feedback (SOF), uses static gains linked to the available states to control the system. The latter strategy leads to a low-cost control design with relatively simple practical implementation since no additional sensors are needed (Dong and Yang, 2008).

The switched control is a strategy that has proven to be efficient, not only in stabilizing, but also in improving the transient response of dynamic systems (Sun and Ge, 2005), which consist in the design of a set of controllers and a switching law such that the closed-loop system is asymptotically stable when the controllers are coordinated under the switching law (Xiao et al., 2020). This particular control strategy has been attracting the interest of the research community, yielding interesting studies such as Lin and Antsaklis (2009) which presents analysis and stabilization strategies for switched linear systems, and a method that uses a common quadratic Lyapunov function. Additionally, quadratic stability to design state feedback controllers can be found in de Souza et al. (2013). However, in this work we will consider the practical engineering problem mentioned before where not all states are available, resulting in the design of switched/hybrid controllers via output feedback (Yang et al. (2015); He et al. (2019); He et al. (2020);Xiao et al. (2020);Li et al. (2016); Carniato et al. (2020); de Oliveira et al. (2018); de Oliveira et al. (2014)).

Considering the problem of output feedback switched control, the literature on switched SOF is scarce, and the available methods usually address the issue through the DOF control perspective, which seeks the benefits of switched control by creating several dynamic gains or observers via output feedback and a switching law for choosing the appropriate controller (Yang et al., 2015). For instance, we can mention a switched DOF controller applied to a highly maneuverable technology vehicle that can be found in He et al. (2019), which demonstrates a smooth-switching linear parameter-varying dynamic output feedback control, a DOF-switched controller that combines input covariance constraint (ICC) that avoids sudden variations between controllers and also minimizes $\mathcal{H}_\infty$ and $\mathcal{H}_2$ cost in the switching law, using parametric linear matrix inequalities (PLMIs), and He et al. (2020) which applies the smooth-switching linear parameter-varying dynamic output feedback control for the vibration reduction of a flexible wing of an airplane. Additionally, in Xiao et al. (2020) one can find two methods for synthesizing switched static output feedback controllers using LMIs obtained through Finsler's Lemma and another using a transformation matrix. Carniato et al. (2020) proposed the use of a hybrid metaheuristic technique, called DE–LMI (differential evolution – linear matrix inequality) to continuous-time uncertain switched linear systems and Bocca et al. (2022) propose a designing procedure for robust guaranteed cost switched SOF also making use of a algorithm DE-LMI to find the desired switched SOF controllers. Considering the pre-

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sented scope, this paper investigates the proposition of new LMI conditions for the design of switched controllers via static output feedback for linear time-invariant (LTI) systems with polytopic uncertainties. In this paper, the LMI framework is used to describe the proposed method, as it consists of a powerful tool to solve control and optimization problems (Scherer et al., 1997), and which can be easily programmed with the MATLAB® in interfaces such as the YALMIP (Yet Another LMI Parser) (Lofberg, 2004) and solved with SeDuMi (Sturm, 1998). The proposed results are inspired by the works of Manesco (2013) and Sereni et al. (2020) that use the two-stage method to solve the SOF problem (Agulhari et al. (2010); Mehdi et al. (2004)), which consists in developing a state feedback gain K, then using this information as input for calculating the desired SOF gains. This approach represents a new alternative for desining robust SOF switched controllers, in contrast to the DE-LMI approach presented in Carniato et al. (2020) and Bocca et al. (2022). The proposed method is obtained using Finsler's Lemma. The switched control design technique via static output feedback is applied on an active suspension system to test and demonstrate the efficacy of our method.

2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider a linear time-invatiant (LTI) system:

$$\begin{aligned} \dot{x}(t) &= A(\alpha)x(t) + B(\alpha)u(t) \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the measured output vector, and $u(t) \in \mathbb{R}^m$ is the control input vector. Moreover, the plant matrix $A(\alpha) \in \mathbb{R}^{n\times n}$ and the input matrix $B(\alpha) \in \mathbb{R}^{n\times m}$ are uncertain matrices that describe the system dynamics, and can be represented in the polytopic domain $\mathfrak{D}$ defined as:

$$\mathfrak{D} = \{(A,B)(\alpha) : (A,B)(\alpha) = \sum_{i=1}^{N} \alpha_i (A,B)_i, \alpha \in \wedge_N\}, \tag{2}$$

where A_i and B_i denote the i-th polytope vertex, and N is the number of vertices of the polytope. Furthermore, $\mathfrak{D}$ is parameterized in terms of a vector $\alpha = (\alpha_1, ..., \alpha_N)$, whose parameters α_i are unknown constants belonging to the unitary simplex set $\wedge_N$, defined as

$$\wedge_N = \{\alpha \in \mathbb{R}^n : \sum_{i=1}^{N} \alpha_i = 1; \alpha_i \geq 0\}, \tag{3}$$

for $i \in \mathbb{K}_N$, where $\mathbb{K}_N$ is a set of positive integers $\{1,\ldots,N\}$. Supposing that the feedback loop is composed by the following control law, presented by Mainardi Júnior et al. (2015)

$$u(t) = L_\sigma y(t), \tag{4}$$

where $\sigma(t)$ is the switching strategy defined by

$$\sigma(t) = \arg\min_{j\in\mathbb{K}_N} (y'(t)Q_j y(t)) = \arg\min_{j\in\mathbb{K}_N} (x'(t)C'Q_jCx(t)) \tag{5}$$

where Q_j are switching matrices. Considering a set of constant gains $L_j \in \mathbb{R}^{m\times p}$, $\forall j \in \mathbb{K}_N$, then the switched controller L_σ, is such that

$$L_\sigma \in \{L_1, L_2, \ldots, L_N\}, \tag{6}$$

and the system (1) in closed-loop assumes the form

$$\dot{x}(t) = [A(\alpha) + B(\alpha)L_\sigma C]x(t). \tag{7}$$

In these terms, the objective is to find L_j and Q_j, $\forall j \in \mathbb{K}_N$, such that when the gain L_σ is selected according to (5), it asymptotically stabilizes (7).

The Finsler's Lemma is a crucial math tool for the approach proposed in this work to solve the control problem via output feedback. Therefore, before properly addressing the issue, the Finsler's Lemma is formally presented in the following sequence

Lemma 1. (Finsler's Lemma) Consider $\mathcal{W} \in \mathbb{R}^{2n}$, $\mathcal{S} \in \mathbb{R}^{2n\times 2n}$ and $\mathcal{R} \in \mathbb{R}^{n\times 2n}$ with $(\text{rank}(\mathcal{R}) < n)$ where $\mathcal{R}^{\perp}$ is a basis for the null space of $\mathcal{R}$ $(i.e.\, \mathcal{R}\mathcal{R}^{\perp} = 0)$.

Then, the following conditions are equivalent:

(i) $\mathcal{W}'\mathcal{S}\mathcal{W} < 0, \forall \mathcal{W} \neq 0, \mathcal{R}\mathcal{W} = 0,$

(ii) $\mathcal{R}'^{\perp}\mathcal{S}\mathcal{R}^{\perp} < 0,$

(iii) $\exists \eta \in \mathbb{R} : \mathcal{S} - \eta\mathcal{R}'\mathcal{R} < 0,$

(iv) $\exists \mathcal{X} \in \mathbb{R}^{2n\times n} : \mathcal{S} + \mathcal{X}\mathcal{R} + \mathcal{R}'\mathcal{X}' < 0$

where η and $\mathcal{X}$ are additional variables (or multipliers).

Proof. See Skelton et al. (1997) and de Oliveira and Skelton (2001)

The transient performance may also be a requirement for some systems. If that is the case, for ensuring such control requirements we might consider the minimum decay rate, which is an index associated with the response speed, defined according to Boyd et al. (1994), as the highest δ such that

$$\lim_{t\to\infty} e^{\delta t}||x(t)|| = 0 \tag{8}$$

holds for all trajectories of $x(t)$ the system's states.

Taking into account the Lyapunov's function $V(x(t)) = x'(t)Px(t)$, a low bound γ on the minimum decay rate can be established if

$$\dot{V}(x(t)) \leq -2\gamma V(x(t)) \tag{9}$$

holds for all trajectories of the system's states x(t), with $\delta > \gamma > 0$ (Boyd et al., 1994).

3. DESIGN OF SWITCHED OUTPUT FEEDBACK CONTROLLERS

The developed studies and proposed contributions in this work for the design of switched output feedback controllers, is based upon the strategy presented in Sereni et al. (2020), Manesco (2013) and in Mehdi et al. (2004), that consists of a two-stage control design. In this strategy we first design a state feedback controller, and then use the derived gain matrix as an input parameter to obtain the desired SOF controller. Considering the control law in the first stage as

$$u(t) = Kx(t), \tag{10}$$

then the system (1) in closed-loop is represented by

$$\dot{x}(t) = [A(\alpha) + B(\alpha)K]x(t). \tag{11}$$

To obtain a robust state feedback gain K that asymptotically stabilizes the system (11), we might consider the quadratic stability condition presented in Boyd et al.

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(1994), which states that if there are matrices $W \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{n\times m}$, and a positive scalar β such that

$$\begin{aligned} &W = W' > 0,\\ &A_iW + WA_i' + B_iZ + Z'B_i' + 2\beta W < 0, \end{aligned} \tag{12}$$

for $\forall i \in \mathbb{K}_N$, then K is given by $K = ZW^{-1}$, and (11) has its minimum decay rate bounded by β.

Remark 1. The quadratic stability condition (12) compose the strategy chosen for the first stage, but it is important to emphasise that as each design stage is performed separately, any other state feedback control design can be implemented to derive the first stage gain matrix K.

Therefore, the gain matrix K obtained in he first step is used as an input parameter for the design of the second step in the Theorem 1, where new sufficient LMI conditions for computing the desired switched SOF controller, regarding the specification of the minimum decay rate, $\gamma > 0$ are proposed.

Theorem *1.* Assuming that there exists a state feedback gain K, such that $A(\alpha) + B(\alpha)K$ is asymptotically stable, with a minimum decay rate greater or equal to $\gamma > 0$, then there exists a stabilizing switched static output feedback controller, L_σ , such that $A(\alpha) + B(\alpha)L_\sigma C$, is asymptotically stable considering the switching rule (5), if there exist symmetric matrices, P, $Q_{0i} \in \mathbb{R}^{n\times n}$ and $Q_j \in \mathbb{R}^{p\times p}$, with

$$P > 0 \tag{13}$$

and matrices F, G, H, and J_j such that

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK + 2\gamma P - Q_{0i} - C'Q_jC & * & * \\ P - F' + GA_i + GB_iK & -G - G' & * \\ B_i'F' + J_jC - HK & B_i'G' & -H - H' \end{bmatrix} < 0 \tag{14}$$

and

$$Q_{0i} + C'Q_iC < 0 \tag{15}$$

for $\forall\, i$ and $j \in \mathbb{K}_N$.

In the affirmative case, the switched output feedback gains are given by $L_j = H^{-1}J_j$, $\forall j \in \mathbb{K}_N$.

Proof. Assuming that (13), (14) and (15) hold, we can see that H is invertible, since according to Boyd et al. (1994), a non-symmetric matrix M is invertible, if $M + M' < 0$.

Now, multiplying (14) and (15) by α_i and considering the switching law (5), we have

$$\begin{bmatrix} \sum_{i=1}^{N}\alpha_i(A_i'F' + FA_i + K'B_i'F' + FB_iK + 2\gamma P - Q_{0i} - C'Q_\sigma C) & * & * \\ \sum_{i=1}^{N}\alpha_i(P - F' + GA_i + GB_iK) & \sum_{i=1}^{N}\alpha_i(-G - G') & * \\ \sum_{i=1}^{N}\alpha_i(B_i'F' + J_\sigma C - HK) & \sum_{i=1}^{N}\alpha_i(B_i'G') & \sum_{i=1}^{N}\alpha_i(-H - H') \end{bmatrix} < 0 \tag{16}$$

$$\sum_{i=1}^{N}\alpha_i(Q_{0i} + C'Q_iC) < 0 \tag{17}$$

Expanding the terms in (16) and (17) and regarding that $\sum_{i=1}^{N}\alpha_i = 1$, we obtain

$$\begin{bmatrix} A'(\alpha)F' + FA(\alpha) + K'B'(\alpha)F' + FB(\alpha)K + 2\gamma P - Q_0(\alpha) - C'Q_\sigma C & * & * \\ P - F' + GA(\alpha) + GB(\alpha)K & -G - G' & * \\ B'(\alpha)F' + J_\sigma C - HK & B'(\alpha)G' & -H - H' \end{bmatrix} < 0 \tag{18}$$

and

$$Q_0(\alpha) + C'Q(\alpha)C < 0. \tag{19}$$

Using the idea presented by Mehdi et al. (2004), pre- and post-multiplying (18) by T_σ and T_σ', where T_σ is

$$T_\sigma = \begin{bmatrix} I & 0 & S_\sigma' \\ 0 & I & 0 \end{bmatrix} \tag{20}$$

it follows that

$$\begin{bmatrix} \psi_\sigma(\alpha) & \phi_\sigma(\alpha) \\ * & -G - G' \end{bmatrix} < 0, \tag{21}$$

where

$$\begin{aligned} \psi_\sigma(\alpha) = {}& A'(\alpha)F' + K'B'(\alpha)F' + S_\sigma'B'(\alpha)F' \\ &+ S_\sigma'J_\sigma C - S_\sigma'HK + FA(\alpha) + FB(\alpha)K \\ &+ FB(\alpha)S_\sigma + C'J_\sigma'S_\sigma - K'H'S_\sigma - Q_0(\alpha) \\ &+ 2\gamma P - C'Q_\sigma C + S_\sigma'(-H - H')S_\sigma \end{aligned} \tag{22}$$

and,

$$\phi_\sigma(\alpha) = P - F + A'(\alpha)G' + K'B'(\alpha)G' + S_\sigma'B'(\alpha)G'. \tag{23}$$

Replacing $S_\sigma = H^{-1}J_\sigma C - K$, in (22) and (23), then $\psi(\alpha)$ and $\phi_\sigma(\alpha)$ can be rewritten as

$$\begin{aligned} \psi(\alpha) = {}& A'(\alpha)F' + K'B'(\alpha)F' - C'Q_\sigma C \\ &+ 2\gamma P + (C'J_\sigma'H'^{-1} - K')B'(\alpha)F' + FA(\alpha) \\ &+ (C'J_\sigma'H'^{-1} - K')J_\sigma C - (C'J_\sigma'H'^{-1} - K')HK \\ &+ FB(\alpha)K + FB(\alpha)(H^{-1}J_\sigma C - K) - Q_0(\alpha) \\ &+ C'J_\sigma'(H^{-1}J_\sigma C - K) - K'H'(H^{-1}J_\sigma C - K) \\ &+ (C'J_\sigma'H'^{-1} - K')(-H - H')(H^{-1}J_\sigma C - K) \end{aligned} \tag{24}$$

and

$$\begin{aligned} \phi_\sigma(\alpha) = {}& P - F + A'(\alpha)G' + K'B'(\alpha)G' \\ &+ (C'J_\sigma'H'^{-1} - K')B'(\alpha)G'. \end{aligned} \tag{25}$$

Expanding the products in (24)

$$\begin{aligned}\psi_\sigma(\alpha) = A'(\alpha)F' + C'J_\sigma' H'^{-1}B'(\alpha)F' \\ +2\gamma P + C'J_\sigma' H'^{-1}J_\sigma C - K'J_\sigma C - Q_0(\alpha) \\ -C'J_\sigma' H'^{-1}HK + FA(\alpha) + FB(\alpha)H^{-1}J_\sigma C \\ +C'J_\sigma' H^{-1}J_\sigma C - C'J_\sigma' K - K'H'H^{-1}J_\sigma C \\ -C'Q_\sigma C - C'J_\sigma' H'^{-1}HH^{-1}J_\sigma C \\ +C'J_\sigma' H'^{-1}HK - C'J_\sigma' H'^{-1}H'H^{-1}J_\sigma C \\ +C'J_\sigma' H'^{-1}H'K + K'HH^{-1}J_\sigma C \\ +K'H'H^{-1}J_\sigma C.\end{aligned} \tag{26}$$

Considering the following equivalent relation

$$H^{-1}H = HH^{-1} = I = H'^{-1}H' = H'H'^{-1}, \tag{27}$$

then, (26) assumes the form below

$$\begin{aligned}\psi_\sigma(\alpha) = (A(\alpha) + B(\alpha)H^{-1}J_\sigma C)'F' \\ +2\gamma P + F(A(\alpha) + B(\alpha)H^{-1}J_\sigma C) \\ -Q_0(\alpha) - C'Q_\sigma C.\end{aligned} \tag{28}$$

Now, making $L_\sigma = H^{-1}J_\sigma$ in (25) and (28), we obtain

$$\begin{aligned}\psi_\sigma(\alpha) = (A(\alpha) + B(\alpha)L_\sigma C)'F' \\ +2\gamma P + F(A(\alpha) + B(\alpha)L_\sigma C) \\ -Q_0(\alpha) - C'Q_\sigma C\end{aligned} \tag{29}$$

and,

$$\phi_\sigma(\alpha) = P - F + (A(\alpha) + B(\alpha)L_\sigma C)'G'. \tag{30}$$

Considering (29) and (30), we can rewrite (21) in terms of a sum of matrices as follows:

$$\begin{bmatrix} 2\gamma P - Q_0(\alpha) - C'Q_\sigma C + F(A(\alpha) + B(\alpha)L_\sigma C) & P - F \\ P + G(A(\alpha) + B(\alpha)CL_\sigma) & -G \end{bmatrix} + \begin{bmatrix} (A(\alpha) + B(\alpha)L_\sigma C)'F' & (A(\alpha) + B(\alpha)CL_\sigma)'G' \\ -F' & -G' \end{bmatrix} < 0 \tag{31}$$

then, splitting the first matrix in (31), and rearranging properly, we have

$$\begin{aligned}\begin{bmatrix} 2\gamma P - Q_0(\alpha) - C'Q_\sigma C & P \\ P & 0 \end{bmatrix} \\ + \begin{bmatrix} F \\ G \end{bmatrix} [(A(\alpha) + B(\alpha)L_\sigma C) \;\; -I] \\ + \begin{bmatrix} (A(\alpha) + B(\alpha)L_\sigma C)' \\ -I \end{bmatrix} \begin{bmatrix} F' & G' \end{bmatrix} < 0.\end{aligned} \tag{32}$$

Considering the following definitions:

$$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} = \mathcal{X}, \tag{33}$$

$$\mathcal{S}_\sigma(\alpha) = \begin{bmatrix} 2\gamma P - Q_0(\alpha) - C'Q_\sigma C & P \\ P & 0 \end{bmatrix} \tag{34}$$

and

$$\mathcal{R}_\sigma(\alpha) = [(A(\alpha) + B(\alpha)L_\sigma C) \;\;\; -I] \tag{35}$$

we can rewrite (32) as

$$\mathcal{S}_\sigma(\alpha) + \mathcal{X}\mathcal{R}_\sigma(\alpha) + \mathcal{R}_\sigma'(\alpha)\mathcal{X}' < 0. \tag{36}$$

Futhermore, note that (36) corresponds to the condition, (iv), of Finsler's Lemma as stated on Lemma 1. Thus, considering the condition, (i), of Finsler's Lemma, which is $\mathcal{W}'\mathcal{S}_\sigma(\alpha)\mathcal{W} < 0$, $\forall \mathcal{W} \neq 0$, $\mathcal{R}_\sigma(\alpha)\mathcal{W} = 0$, and assuming that $\mathcal{W} = \begin{bmatrix} x'(t) & \dot{x}'(t) \end{bmatrix}'$, we can derive regarding (35), (34) and (33) the following expressions:

$$[(A(\alpha) + B(\alpha)L_\sigma C) \;\; -I] \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = 0 \tag{37}$$

$$\begin{bmatrix} x'(t) & \dot{x}'(t) \end{bmatrix} \begin{bmatrix} 2\gamma P - Q_0(\alpha) - C'Q_\sigma C & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} < 0. \tag{38}$$

Therefore, according to (37) we have

$$\dot{x}(t) = (A(\alpha) + B(\alpha)L_\sigma C)x(t), \tag{39}$$

which corresponds to the closed-loop system equation (7).

Furthermore, (38) leads to

$$\begin{aligned}\dot{x}'(t)Px(t) + x'(t)P\dot{x}(t) < x'(t)(Q_0(\alpha) \\ +C'Q_\sigma C - 2\gamma P)x(t).\end{aligned} \tag{40}$$

Making $V(x(t)) = x'(t)Px(t)$ and rearranging, we can conclude that (40) becomes

$$\dot{V}(x(t)) + 2\gamma V(x(t)) < x'(t)(Q_0(\alpha) + C'Q_\sigma C)x(t). \tag{41}$$

Pre- and post-multiplying (19) by $x'(t)$ and $x(t)$, we have by initial assumption that

$$x'(t)(Q_0(\alpha) + C'Q(\alpha)C)x(t) < 0. \tag{42}$$

As shown in de Souza et al. (2013), the minimum of a set of real numbers is less than or equal to an arbitrary convex combination of these numbers. Therefore, we have that

$$\begin{aligned}x'(t)(Q_0(\alpha) + C'Q_\sigma C)x(t) = \min_{\forall j \in \mathbb{K}_N} (x'(t)(Q_0(\alpha) \\ +C'Q_jC)x(t)) \leq x'(t)(Q_0(\alpha) + C'Q(\alpha)C)x(t) < 0\end{aligned} \tag{43}$$

then we can affirm that

$$x'(t)(Q_0(\alpha) + C'Q_\sigma C)x(t) < 0 \tag{44}$$

so, finally, we can conclude that

$$\dot{V}(x(t)) < -2\gamma V(x(t)) \tag{45}$$

which is Lyapunov's equation for stability, considering the minimum decay rate as defined in (9) (Boyd et al., 1994).

Remark 2. For simplicity, and without loss of generality, the number of switching modes are considered to be equal to the number of vertices of the uncertainty polytope.

4. ANALYSIS

Intending to investigate the efficacy of the presented method, the results of application in a numerical and a practical systems are presented and discussed in this section.

Remark 3. Due to the relationship between the size of the output matrix C and the size of the Q switching matrices, the number of outputs available for static output feedback must be considered for controller design, In fact, observe that if $p = 1$, the only matrix Q_i possible to be chosen will be

$$\min_{\forall\, i \in \mathbb{K}_N} Q_i. \tag{46}$$

4.1 Academic Example

Experiment *1.* In this numerical experiment an uncertain system described according to (1) is considered. This system can be represented in terms of convex combination of the following vertices

- Vertex 1

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ -10 & -30 & -15 \\ 0 & 6 & -1 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (47)$$

- Vertex 2

$$A_2 = \begin{bmatrix} 1 & 0 & 1 \\ -10 & -50 & -15 \\ 0 & 6 & -1 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (48)$$

- Vertex 3

$$A_3 = \begin{bmatrix} 1 & 0 & 1 \\ -10 & -30 & -10 \\ 0 & 6 & -1 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (49)$$

- Vertex 4

$$A_4 = \begin{bmatrix} 1 & 0 & 1 \\ -10 & -50 & -10 \\ 0 & 6 & -1 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (50)$$

where the state vector in the numerical system as

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}. \quad (51)$$

Programming the LMI (12) via MATLAB® software, and solving via YALMIP interface (Lofberg, 2004) and the solver SeDuMi, and fixing the decay rate bounds of the first and second stage of the project as $\beta = \gamma = 0.5$, we obtain the state feedback gain

$$K = [-31.426 \; 96.175 \; 17.324]\,. \quad (52)$$

And, in the second stage, using the strategy of Robust SOF proposed by Manesco (2013) and the first-stage state feedback gain (52), the designed static output feedback gain is

$$L_{Manesco} = [-69.317 \; 16.449]\,. \quad (53)$$

On its turn, using the LMIs proposed in Theorem 1 and the first-stage state feedback gain (52), the designed switched static output feedback gains are

$$\begin{aligned} L_1 &= [-88.41273 \; -3.89826] \\ L_2 &= [-88.52035 \; -4.09084] \\ L_3 &= [-88.47736 \; -3.39742] \\ L_4 &= [-88.44582 \; -3.3022] \end{aligned} \quad (54)$$

with the switching matrices Q_1, Q_2, Q_3 and Q_4:

$$\begin{aligned} Q_1 &= \begin{bmatrix} -2.3476 \times 10^9 & -2.597 \times 10^9 \\ -2.597 \times 10^9 & 5.4722 \times 10^8 \end{bmatrix} \\ Q_2 &= \begin{bmatrix} -2.6677 \times 10^9 & 2.0292 \times 10^8 \\ 2.0292 \times 10^8 & 3.7016 \times 10^8 \end{bmatrix} \\ Q_3 &= \begin{bmatrix} -1.4916 \times 10^9 & -2.1894 \times 10^9 \\ -2.1894 \times 10^9 & -2.8384 \times 10^9 \end{bmatrix} \\ Q_4 &= \begin{bmatrix} -2.3395 \times 10^9 & 2.4527 \times 10^9 \\ 2.4527 \times 10^9 & -1.5759 \times 10^9 \end{bmatrix}. \end{aligned} \quad (55)$$

The system was simulated in Simulink® with the initial state conditions defined as

$$x(0) = \begin{bmatrix} 10 \\ 1 \\ 1 \end{bmatrix} \quad (56)$$

and the results with the designed controllers are shown in Figure 1. For the experiment, we set $\alpha_i = 0.25$ for all $i \in \mathbb{K}_N$, in order to visualize the closed-loop behavior.

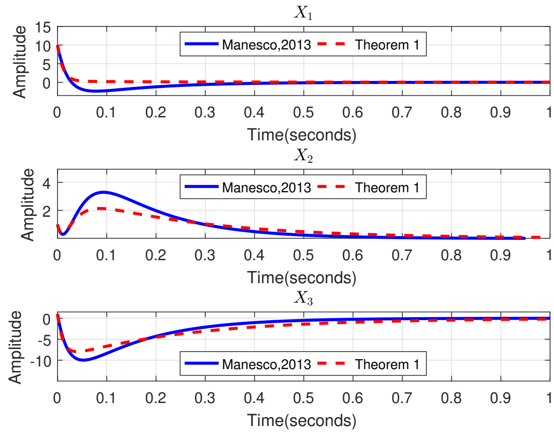

Figure 1. System behavior with Switched Control via Output Feedback and robust SOF controller Manesco (2013).

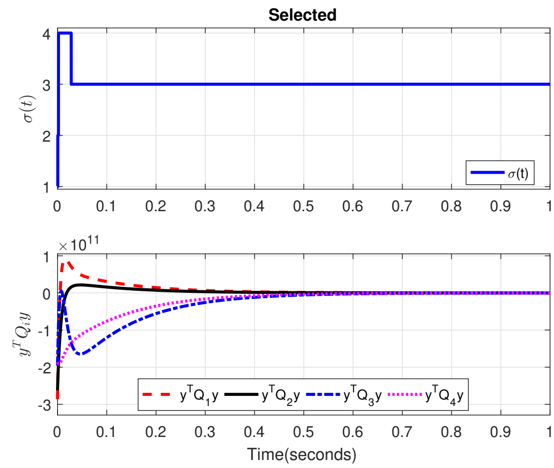

Figure 2. Switching law and the value of each $y'Q_iy$ in the course of the simulation.

It can be observed that the switched controller proposed in this work has a better response, as the system state variables have a faster convergence to the origin. Figure 2 illustrates the behavior of the switching law, which changes the selected controller according to the instantaneous value of each $y'Q_iy$ in the course of the simulation.

4.2 Practical Implementation in a Bench-Scale Active Suspension System

The Active Suspension System represents a quarter-car model, formed by three parts: the vehicle body, that is suspended over the tire assembly by springs and the active suspension mechanism. The tire assembly is in contact with the tire through springs, and the tire has to be able to pass through different types of terrain without compromising the passenger's comfort. In the performed experiment the equipment used is a Quanser® Active

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Suspension system as shown in Figure 3, in which the parts of the car are represent by plates, or floors, and the active suspension mechanism is emulated by a DC motor, and the road profiles are simulated by another motor DC (Quanser, 2009). In Figure 4 the schematic diagram of the

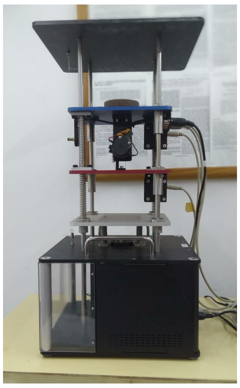
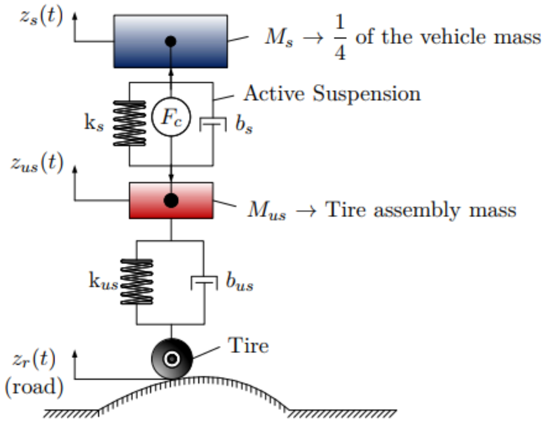


Figure 3. Quanser® Active Suspension. Property of the Laboratory of Research in Control at FEIS-UNESP.

system is illustrated, where M_s is a representation of 1/4 of the vehicle body mass, M_{us} is as the mass of the tire, and k_s, k_{us}, b_s and b_{us}, are the springs and dampers in the model assembly. $z_s(t)$ and $z_{us}(t)$ are the related position of the body floor and tire assembly floor. Finally, $z_r(t)$ is the input of the system, which represents the surface profile of the road, and $F_c(t)$ is the active suspension control command.



Source: Adapted from Silva (2012).

Figure 4. Schematic diagram of an active suspension system.

The system illustrated in the Figure 4 can be described in a state-space model presented by Quanser (2009) as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{-k_s}{M_s} & \frac{-b_s}{M_s} & 0 & \frac{b_s}{M_s} \\ 0 & 0 & 0 & -1 \\ \frac{k_s}{M_{us}} & \frac{b_s}{M_{us}} & \frac{-k_{us}}{M_{us}} & \frac{-(b_s+b_{us})}{M_{us}} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{\rho}{M_s} \\ 0 \\ \frac{-\rho}{M_{us}} \end{bmatrix} u(t),$$
$$y(t) = Cx(t), \tag{57}$$

where $0 < \rho \leq 1$ is an uncertain parameter that acts as a possible fault in the actuator. Furthermore, the state and the input vectors in (57) are defined as

$$\dot{x}(t) = \begin{bmatrix} z_s(t) - z_{us}(t) \\ \dot{z}_s(t) \\ z_{us}(t) - z_r(t) \\ \dot{z}_{us}(t) \end{bmatrix} \text{ and } u(t) = F_c(t). \tag{58}$$

The parameters presented in Table 1 are considered for the experiments (Quanser, 2009).

Table 1. Active Suspension Parameters.

Parameter	Value	Parameter	Value
M_s	2.45 Kg	M_{us}	1.0 Kg
k_s	900 N/m	k_{us}	2500 N/m
b_s	7.5 Ns/m	b_{us}	5.0 Ns/m

Source: Adapted from Quanser (2009).

Experiment *2.* Intending to compare the performance of the Theorem 1 and the robust static output feedback strategy proposed by Manesco (2013), in this experiment the Active Suspension System is considered to present a fault of up to 25% power loss (*i.e.* $0.75 \leq \rho \leq 1$). Also, it is supposed that only the measurement of state variable $(z_s(t) - z_{us}(t))$ and $\dot{z}_s(t)$ are available, the active suspension may be described in terms of a polytope with two vertices:

- Vertex 1 (without fault)

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.347 & -3.061 & 0 & 3.061 \\ 0 & 0 & 0 & -1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0 \\ 0.408 \\ 0 \\ -1 \end{bmatrix} \tag{59}$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- Vertex 2 (with fault - 25% of power loss)

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.347 & -3.061 & 0 & 3.061 \\ 0 & 0 & 0 & -1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 0 \\ 0.306 \\ 0 \\ -0.75 \end{bmatrix} \tag{60}$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The LMI (12) was solved similarly as in Experiment 1, fixing the decay rate bounds of the first and second stage of the project as $\beta = \gamma = 0.75$, we obtain the state feedback gain

$$K = [-132.74 \ -4.237 \ 317.92 \ 1.4773]. \tag{61}$$

And, in the second stage, using the strategy of robust SOF proposed by Manesco (2013) and the same first-stage state feedback gain (61), the designed static output feedback gain is

$$L_{Manesco} = [-101.43 \ -23.359]. \tag{62}$$

Finally, in the second stage, using the LMIs in Theorem 1, and the first-stage state feedback gain (61), the designed switched static output feedback gains are

$$L_1 = [-129.22894 - 36.78596]$$
$$L_2 = [-128.43995 - 36.65751] \tag{63}$$

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with the switching matrices Q_1 and Q_2:

$$Q_1 = \begin{bmatrix} -1.8325 \times 10^{-16} & -2.2453 \times 10^{-16} \\ -2.2453 \times 10^{-16} & -7.2209 \times 10^{-17} \end{bmatrix}$$
$$Q_2 = \begin{bmatrix} -0.029572 & -0.010624 \\ -0.010624 & -0.0014828 \end{bmatrix}. \quad (64)$$

According to Figure 5 the designed switched controller offered an improvement when compared with the robust static output feedback control (Manesco, 2013), which can be seen in terms of the suppression of the oscillations in the system, even in the event of failure, imposed after $t = 8s$, the superior performance prevails.

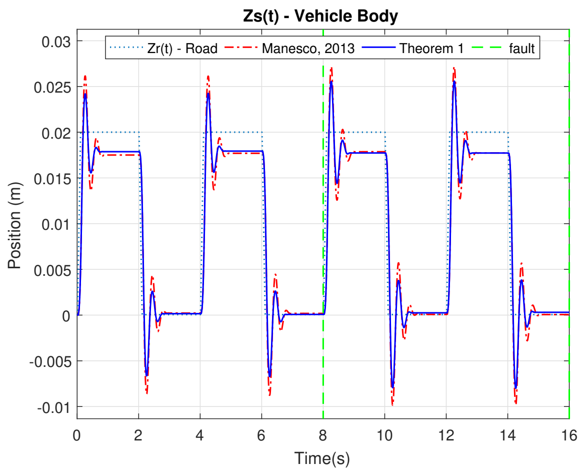

Figure 5. Active Suspension System behavior with Switched Control via Output Feedback and robust SOF controller Manesco (2013) closed-loop (0-8s); fault: 25% power loss in the actuator (8-16s).

Figure 2 illustrates the switching law behavior and the value of each $y'Q_iy$ in the experiment, and comparing with experiment 1, we can observe that states with a more oscillatory behavior causes a switching frequency more accentuated, as seen in this case.

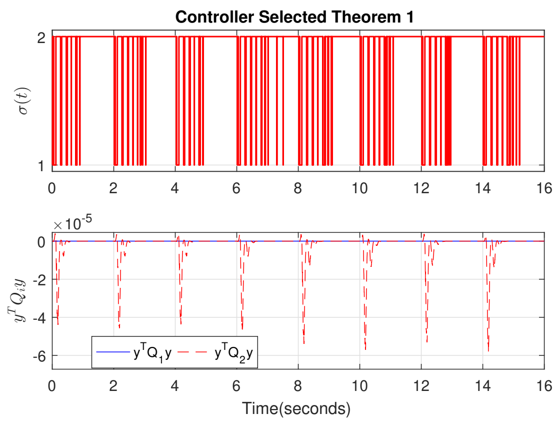

Figure 6. Controller switching in Experiment 2 and the value of each $y'Q_iy$ in the course of the simulation.

5. CONCLUSION

In this paper, we proposed a new a design strategy for switched static output feedback controllers, based on the two-stage control design to the best of the authors' knowledge, this present approach has not been addressed in the literature so far. We also considered the specification of the minimum decay rate in both stages for an improvement in the dynamic response of the system. The results of a practical implementation of a SOF controller designed via Theorem 1 for the Quanser® Active Suspension showed that the controller was able to suppress the oscillations on the suspension, even during the occurrence of a fault on the system actuator, and the switched controller had a better performance when compared to a robust SOF design. Future studies on this subject include a deeper comparison analysis on performance and feasibility with similar approaches. Less conservative LMIs in terms of parameter-dependent matrices P, F and G are also a direct extension of these preliminary results.

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