Robust Regional Input-to-State Stabilization of Discrete-Time Systems under Magnitude and Rate Saturating Actuators ⋆

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Abstract: We propose new convex conditions, formulated in terms of linear matrix inequalities, allowing the design of robust state feedback controllers for time-varying discrete-time systems under saturating actuators. We consider the case where the actuators can be saturated on both magnitude and rate, which leads to more complex but realistic conditions. Because of the nonlinearities, the regional input-to-state stability is employed, handling energy-bounded disturbance signals. The feasibility of the proposed conditions allows computing safe estimates of the initial conditions sets depending on the disturbance energy. Two numerical examples illustrate the proposal's application and serve to establish numerical comparisons with other approaches from the literature.

Keywords: Robust stabilization, Input-to-state stability, Saturating actuators, Rate saturation, Magnitude saturation, Discrete-time systems, Regional stability, Time-varying systems.

1. INTRODUCTION

Limitations in the actuators are a constant concern in practical control systems, which has been thoroughly studied in the past years. Practical applications often require magnitude or rate constraints on actuators due to operational security conditions, available energy, or physical limitations. Such constraints may degenerate performance, include spurious equilibrium points, and even yield unstable behavior (Tarbouriech et al., 2011, pp. 5). Although most of the literature's works only discuss magnitude constraints, the neglected effects of actuators rate saturation may cause severe accidents, like the meltdown that occurred in the Chernobyl power plant (Stein, 2003) or pilot-induced oscillation (PIO) resulting in aircraft crashes (Klyde et al., 1997; Duda, 1997).

Two main approaches can be identified among the few works dealing with both magnitude and rate stabilization. In the first one, the controller is designed to embed the nonlinear actuator model, not allowing the control signal to violate the magnitude and rate limits of the actuator (Kapila et al., 1999; Pan and Kapila, 2001; Gomes da Silva Jr. et al., 2007). In the second approach, a first-order system with a saturating input signal (describing the magnitude saturation) and the rate saturation modeled as the actuator state's constraint is employed, leading to a particular case of nested saturations (Tarbouriech et al., 2006; Bateman and Lin, 2003; Zhou, 2013). Such an approach is also used in this work and has been recently employed by Palmeira et al. (2016) to address the stability analysis of magnitude and rate saturating sampled-data control in the continuous-time framework. Oliveira et al. (2022) use the second approach to explore regional polyquadratic stabilization for discrete-time LPV systems. However, even some works dealing with time-varying systems, none of them cover uncertain systems subject to disturbances in the discrete-time framework.

In this work, we consider the robust stability of parameter-varying uncertain systems under disturbance perturbations. Among early works, we highlight (Geromel et al., 1991), where a numerically tractable formulation is provided for the robust stability analysis and state feedback control design. The approach uses a quadratic (parameter-independent) Lyapunov candidate function for (uncertain) continuous and discrete time-varying systems. Therefore, some works were developed taking into account the rates of variation of the parameters, aiming to relax the quadratic approach, such as (Kaminer et al., 1993; Peres et al., 1994). However, even if the parameters have limited variation rates, the quadratic stability approach considers the worst-case scenario, i.e., arbitrarily fast variation, which may lead to high conservative solutions. To mitigate the conservatism associated with the parameters' independence, some authors have proposed using parameter-dependent Lyapunov functions. In (Daafouz and Bernussou, 2001),

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the authors propose the robust polyquadratic stability of discrete-time systems regardless of parameter variation in a polytope, certified by an affine parameter-dependent Lyapunov function. Leite and Peres (2004) have addressed the robust stabilization of linear discrete-time systems with uncertain parameters with piecewise Lyapunov matrices. Later, Lee (2006) have investigated the characterization of discrete-time systems with arbitrary time-varying parameters stability increasing a set of LMI conditions through path-dependent Lyapunov functions. Using polynomial homogeneous Lyapunov functions, Oliveira and Peres (2009) have considered the robust stabilization in the discrete-time framework by modeling the time-varying parameters in a polytopic domain and assuming that the bounds on their rate of variation are known. Other advances in robust control in the last decade are the use of Lyapunov functions with non-monotonic terms (Lacerda and Seiler, 2017) and the expression of time-varying parameters in discrete-time as solutions of a linear difference equation (Palma et al., 2020). Other methods relying polyhedral Lyapunov functions can be found for instance on (Ernesto et al., 2021). Despite the great development found in linear parameter varying systems, the robust control approach still plays a relevant role in real-world applications because access to systems' parameters is not always available. Recent works dealing with robust controllers with (magnitude) saturating actuators can be found in (Boeff et al., 2019; Saifia et al., 2020).

The regional stability must be preserved because we are dealing with nonlinear systems. Therefore, considering the presence of exogenous signals is a required task to deliver theoretical results to practical applications. Therefore, an alternative is to define a set of admissible signals limited in energy (Tarbouriech et al., 2011, pp. 26) to which the closed-loop system remains stable. This characterization can be made through the Input-to-State Stability (ISS) design (Sontag, 2008).

The main objective of this work is to develop a convex method to design robust input-to-state stabilizing controllers for discrete-time systems under magnitude and rate saturating actuators and exogenous energy disturbance signals. We consider the class of the above-mentioned saturating uncertain time-varying discrete-time systems under ℓ_2 (quadratic summable) exogenous signals, covering a key theoretical gap in such a class of systems. This work extends the proposal in Oliveira et al. (2022) by handling the exogenous disturbance signal and considering the practical case of robust control, which is useful whenever the parameters are not available for online controller tuning. We formulated the problem with nested saturation functions employing generalized sector conditions and parameter dependent Lyapunov functions, which leads to the well-known Lur'e type system. Two numerical examples are presented to compare our approach with others from the literature and illustrate our proposal's efficacy.

Notation. The set of real numbers is denoted by $\mathbb{R}$, while $M \in \mathbb{R}^{n\times m}$ and $x \in \mathbb{R}^n$ are, respectively, a matrix with dimension $n \times m$ with real entries and the vector with n positions and real entries. The M transpose is represented by $M^\top$ and $M_{(i)}$ (M_{ii}) denotes the i-th row (diagonal element (i,i)) of the matrix M. $M_{(i)}^\top$ stands for $(M_{(i)})^\top$. The Euclidean norm of x is denoted by $\|x\|$. $\text{diag}\{M_1, M_2\}$ is a block diagonal matrix composed by M_1 and M_2. In square matrices, the symbol $\star$ represents the symmetric transposed blocks. The matrices $\mathbf{I}$ and $\mathbf{0}$ denote, respectively, the identity and the null matrices of appropriate dimensions.

2. PROBLEM FORMULATION

Consider the uncertain time-varying discrete-time system subject to magnitude and rate saturating actuators and energy disturbance signals:

$$\begin{aligned} x_{k+1} &= A(\alpha_k)x_k + B(\alpha_k)\phi_k(u_k) + B_w(\alpha_k)\omega_k, \\ y_k &= C(\alpha_k)x_k, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^{n_u}$ is the control signal vector, $y_k \in \mathbb{R}^{n_y}$ is the measurable output signal, $\omega_k \in \mathbb{R}^{n_w}$ is an ℓ_2 signal at time k, i.e., ω_k is a quadratically summable perturbation signal belonging to

$$\mathcal{W} = \{\omega \in \mathbb{R}^{n_w} : \|\omega\|_2^2 \leq \delta^{-1}\}, \tag{2}$$

with $\|\omega\|_2 = \sqrt{\sum_{k=0}^{\infty} \omega_k^\top \omega_k}$ and $\delta^{-1} \in \mathbb{R}_+$ represents the maximum energy of the disturbance signals. The uncertain time-varying matrices $A(\alpha_k) \in \mathbb{R}^{n\times n}$, $B(\alpha_k) \in \mathbb{R}^{n\times n_u}$, $B_w(\alpha_k) \in \mathbb{R}^{n\times n_w}$, and $C(\alpha_k) \in \mathbb{R}^{n_y\times n}$ belong to a polytopic domain given by the convex combination of N known vertices as follows:

$$M(\alpha_k) = \sum_{i=1}^{N} \alpha_{k(i)} M_i, \tag{3}$$

with M replacing matrices A, B, B_w, and C where $\alpha_k \in \mathcal{P}$ is a vector of uncertain time-varying parameters, and belongs to the unit simplex:

$$\mathcal{P} = \left\{\kappa \in \mathbb{R}^N : \sum_{i=1}^{N} \kappa_{(i)} = 1, \kappa_{(i)} \geq 0, i = 1, \ldots, N\right\}. \tag{4}$$

This work does not consider constraints on the variation of α_k.

Figure 1 presents the discrete-time counterpart of the *position-type feedback model with speed limitation*, which models each actuator channel (Tyan and Bernstein, 1997).

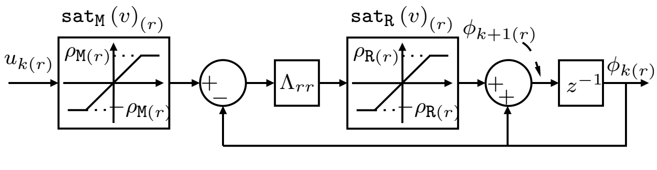

Figure 1. Schematic diagram of the first-order nonlinear discrete-time system used to model magnitude and rate saturation.

Therefore, from such a figure, the output is given by $\phi_k(\cdot) : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ and has the dynamics

$$\bar{x}_{k+1(r)} = \bar{x}_{k(r)} + \texttt{sat}_\texttt{R}\left(\Lambda \texttt{sat}_\texttt{M}(u_k) - \Lambda \bar{x}_k\right)_{(r)}, \tag{5}$$

$$\phi_k(u_k) = \bar{x}_k, \tag{6}$$

for $r = 1, \ldots, n_u$, where $\bar{x}_k \in \mathbb{R}^{n_u}$ is the actuators' state, $\Lambda \in \mathbb{R}^{n_u\times n_u}$ is a diagonal matrix composed by the actuators' poles, and for a signal $v \in \mathbb{R}^{n_u}$ the symmetric saturation functions concerning the rate and magnitude are given by $\texttt{sat}_\texttt{R}(v)_{(r)} = \texttt{sign}(v_{(r)}) \min(|v_{(r)}|, \rho_{\texttt{R}(r)})$, and $\texttt{sat}_{\texttt{M}}(v)_{(r)} = \texttt{sign}(v_{(r)}) \min(|v_{(r)}|, \rho_{\texttt{M}(r)})$, respectively, with $\rho_{\texttt{R}} \in \mathbb{R}^{n_u}$ and $\rho_{\texttt{M}} \in \mathbb{R}^{n_u}$ denoting the respective symmetrical bounds. A state feedback control law, including the actutor's state, is selected to robustly stabilizes the closed-loop system (1)–(6):

$$u_k = \hat{K} z_k = K x_k + \bar{K} \bar{x}_k, \tag{7}$$

thus leading to $z_k = \begin{bmatrix} x_k^\top & \bar{x}_k^\top \end{bmatrix}^\top \in \mathbb{R}^{n+n_u}$, $K \in \mathbb{R}^{n_u \times n}$, $\bar{K} \in \mathbb{R}^{n_u \times n_u}$, and $\hat{K} = \begin{bmatrix} K & \bar{K} \end{bmatrix}$.

Therefore, the closed-loop system (1)–(7), is given by:

$$x_{k+1} = A(\alpha_k) x_k + B(\alpha_k) \phi_k \left(\hat{K} z_k \right) + B_w(\alpha_k) \omega_k. \tag{8}$$

Differently from (Oliveira et al., 2022), we take into account the disturbance signals and assume that the time-varying parameters α_k are not available. Such a difference requires robust controllers and has a practical aspect as exogenous signals are concerned. Define the dead-zone functions to handle the actuators' nonlinearities $\psi_{\texttt{M}k} = \texttt{sat}_{\texttt{M}}(u_k) - u_k$, $\psi_{\texttt{R}k} = \texttt{sat}_{\texttt{R}}(\Lambda \texttt{sat}_{\texttt{M}}(u_k) - \Lambda \bar{x}_k) - (\Lambda \texttt{sat}_{\texttt{M}}(u_k) - \Lambda \bar{x}_k)$, and using the control law (7), $\psi_{\texttt{R}k}$ can be rewritten as $\psi_{\texttt{R}k} = \texttt{sat}_{\texttt{R}} \left(\left(\hat{\Lambda} + \Lambda \hat{K} \right) z_k + \Lambda \psi_{\texttt{M}k} \right) - \left(\left(\hat{\Lambda} + \Lambda \hat{K} \right) z_k + \Lambda \psi_{\texttt{M}k} \right)$, where $\hat{\Lambda} = [\mathbf{0} \;\; -\Lambda]$. Therefore, we can rewrite the closed-loop system as an augmented Lur'e type system as:

$$z_{k+1} = \left(\hat{A}(\alpha_k) + \bar{B} \hat{K} \right) z_k + \begin{bmatrix} \bar{B} & \hat{B} \end{bmatrix} \begin{bmatrix} \psi_{\texttt{M}k} \\ \psi_{\texttt{R}k} \end{bmatrix} + \hat{B}_w(\alpha_k) \omega_k, \tag{9}$$

with $y_k = \hat{C}(\alpha_k) z_k$,

$$\begin{gathered} \hat{A}(\alpha_k) = \begin{bmatrix} A(\alpha_k) & B(\alpha_k) \\ \mathbf{0} & \mathbf{I} - \Lambda \end{bmatrix}, \; \bar{B} = \begin{bmatrix} \mathbf{0} \\ \Lambda \end{bmatrix}, \; \hat{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}, \\ \hat{B}_w(\alpha_k) = \begin{bmatrix} B_w(\alpha_k) \\ \mathbf{0} \end{bmatrix}, \; \hat{C}(\alpha_k) = \begin{bmatrix} C(\alpha_k) \\ \mathbf{0} \end{bmatrix}^\top. \end{gathered} \tag{10}$$

Because the resulting closed-loop system (9) is nonlinear, the regional (or local) stability must be addressed, including the characterization of a region of initial conditions $\mathcal{R}_{\mathcal{A}} \subseteq \mathbb{R}^{n+n_u}$, such that the trajectories initiating in such a region converge asymptotically to the origin. We call $\mathcal{R}_{\mathcal{A}}$ the maximal region of attraction. Furthermore, we need to consider the energy of $\omega \in \mathcal{W}$ with $\delta > 0$, and the set of initial conditions $\mathcal{R}_0 \subseteq \mathcal{R}_{\mathcal{A}}$ must be determined to ensure that the closed-loop trajectories do not leave the region $\mathcal{R}_{\mathcal{A}}$. To guarantee regional stability in the presence of energy-limited exogenous signals, the following definition (Sontag, 2008) is considered.

Definition 1. Consider a positive scalar δ and any sequence $\omega \in \mathcal{W}$. The resulting closed-loop system is said to be regional input-to-state stable (ISS) if for any initial state belonging to $\mathcal{R}_0$ the resulting state trajectories remain bounded in $\mathcal{R}_{\mathcal{A}}$ for all $k \geq 0$. Moreover, if the disturbance is vanishing, then the state trajectories converge towards the origin.

Due to the difficulties in determining the region of attraction, such as non-convexity and eventually not being limited in certain directions (Tarbouriech et al., 2011, pp. 14), we search for an estimate $\mathcal{R}_{\mathcal{E}} \subseteq \mathcal{R}_{\mathcal{A}}$ and $\mathcal{R}_{\mathcal{E}0} \subseteq \mathcal{R}_0$ as large as possible. Therefore, we can describe the problem investigated in this work as follows.

Problem 2. Determine a robust state feedback gain $\hat{K}$ and estimates $\mathcal{R}_{\mathcal{E}0} \subseteq \mathcal{R}_0$ and $\mathcal{R}_{\mathcal{E}} \subseteq \mathcal{R}_{\mathcal{A}}$ such that the closed-loop system (9) is regional ISS for all $\omega \in \mathcal{W}$ and $\alpha_k \in \mathcal{P}$. In addition, the designed controller must ensure a certain upper limit of the ℓ_2-gain, denoted by γ, between the disturbance signal ω and the regulated output y, such that

$$\|y\|_2 = \sqrt{\gamma}(\|\omega\|_2 + \mathfrak{o}), \tag{11}$$

where the bias term $\mathfrak{o}$ is due to the non-null initial conditions.

2.1 Auxiliary Results

The following two lemmas were presented in (Oliveira et al., 2022) as an adaptation of the (Tarbouriech et al., 2006, Lemma 1). These lemmas are used to obtain the main conditions of this work, with the sets:

$$\mathbb{S}_{\texttt{M}}(\rho_{\texttt{M}}) = \left\{ z_k \in \mathbb{R}^{n+n_u} : \left| \left(\left(\hat{K} - G_{\texttt{M}} \right) z_k \right)_{(r)} \right| \leq \rho_{\texttt{M}(r)} \right\}, \tag{12}$$

$$\begin{aligned} \mathbb{S}_{\texttt{R}}(\rho_{\texttt{R}}) = \Big\{ & z_k \in \mathbb{R}^{n+n_u}, \psi_{\texttt{M}k} \in \mathbb{R}^{n_u} : \\ & \left| \left(\left(\begin{bmatrix} \hat{\Lambda} + \Lambda \hat{K} & \Lambda \end{bmatrix} - G_{\texttt{R}} \right) \begin{bmatrix} z_k \\ \psi_{\texttt{M}k} \end{bmatrix} \right)_{(r)} \right| \leq \rho_{\texttt{R}(r)} \Big\}, \end{aligned} \tag{13}$$

for $r = 1, \ldots, n_u$. The set $\mathbb{S}_{\texttt{M}}(\rho_{\texttt{M}})$ concerns the magnitude saturation while the set $\mathbb{S}_{\texttt{R}}(\rho_{\texttt{R}})$ regards the rate saturation, and presents nested saturation function, because the rate saturation function is dependent on the magnitude one.

Lemma 3. If $z_k \in \mathbb{S}_{\texttt{M}}(\rho_{\texttt{M}})$, then the nonlinearity $\psi_{\texttt{M}k}$ satisfies the following inequality:

$$\psi_{\texttt{M}k}^\top T_{\texttt{M}} (\psi_{\texttt{M}k} + G_{\texttt{M}} z_k) \leq 0, \tag{14}$$

for any diagonal matrix $T_{\texttt{M}} > 0$ belonging to $\mathbb{R}^{n_u \times n_u}$.

Lemma 4. If z_k and $\psi_{\texttt{M}k} \in \mathbb{S}_{\texttt{R}}(\rho_{\texttt{R}})$, then the nonlinearity $\psi_{\texttt{R}k}$ satisfies the following inequality:

$$\psi_{\texttt{R}k}^\top T_{\texttt{R}} (\psi_{\texttt{R}k} + G_{\texttt{R}} \begin{bmatrix} z_k^\top & \psi_{\texttt{M}k}^\top \end{bmatrix}^\top) \leq 0, \tag{15}$$

for any diagonal matrix $T_{\texttt{R}} > 0$ belonging to $\mathbb{R}^{n_u \times n_u}$.

For the proof of these lemmas, see (Oliveira et al., 2022).

3. MAIN RESULTS

The following theorem provides a solution to Problem 2 through an efficient numerical feasibility procedure formulated in terms of linear matrix inequalities.

Theorem 5. Consider the uncertain and discrete-time-varying system (1)-(6) under magnitude and rate saturating actuators, and the saturation limits $\rho_{\texttt{M}}$ and $\rho_{\texttt{R}}$ for the actuators' magnitude and rate, respectively. Suppose that there exist symmetric and positive definite matrices $\tilde{P}_i \in \mathbb{R}^{(n+n_u) \times (n+n_u)}$, diagonal matrices $L_{\texttt{M}}$, $L_{\texttt{R}} \in \mathbb{R}^{n_u \times n_u}$, matrices Z, $X_{\texttt{M}}$, $X_{\texttt{R}_1} \in \mathbb{R}^{n_u \times (n+n_u)}$, $X_{\texttt{R}_2} \in \mathbb{R}^{n_u \times n_u}$, $\tilde{H} \in \mathbb{R}^{(n+n_u) \times (n+n_u)}$, i, $j = 1, \ldots, N$, positive scalars μ and δ, with matrices $\hat{A}_i$, $\bar{B}$, $\hat{B}$, $\hat{B}_{wi}$, and $\hat{C}_i$ structured as in (10), such that the following LMIs

$$\begin{bmatrix} \Theta_1 & \Theta_2 \\ \star & \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \star & -\gamma \mathbf{I} \end{bmatrix} \end{bmatrix} < \mathbf{0}, \tag{16}$$

$$\begin{bmatrix} -\tilde{P}_i & Z_{(r)}^\top - X_{\texttt{M}(r)}^\top \\ \star & -\mu \rho_{\texttt{M}(r)}^2 \end{bmatrix} \leq \mathbf{0}, \tag{17}$$

$$\begin{bmatrix} -\tilde{P}_i & X_{\mathtt{M}}^\top & \tilde{H}^\top \hat{\Lambda}_{(r)}^\top + (\Lambda Z)_{(r)}^\top - X_{\mathtt{R}_1(r)}^\top \\ \star & -2L_{\mathtt{M}} & L_{\mathtt{M}}\Lambda_{(r)}^\top - X_{\mathtt{R}_2(r)}^\top \\ \star & \star & -\mu\rho_{\mathtt{R}(r)}^2 \end{bmatrix} \leq \mathbf{0}, \quad (18)$$

and

$$\delta - \mu \geq 0 \quad (19)$$

are feasible for $\forall r = 1, \ldots, n_u$, with

$$\Theta_1 = -\begin{bmatrix} \tilde{P}_i & \tilde{H}^\top \hat{A}_i^\top + Z^\top \bar{B}^\top & X_{\mathtt{M}}^\top & X_{\mathtt{R}_1}^\top \\ \star & -\tilde{P}_j - \tilde{H} - \tilde{H}^\top & \bar{B}L_{\mathtt{M}} & \hat{B}L_{\mathtt{R}} \\ \star & \star & 2L_{\mathtt{M}} & X_{\mathtt{R}_2}^\top \\ \star & \star & \star & 2L_{\mathtt{R}} \end{bmatrix}, \quad (20)$$

$$\Theta_2 = \begin{bmatrix} \mathbf{0} & \tilde{H}^\top \hat{C}_i^\top \\ -\hat{B}_w & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \quad (21)$$

Then, the robust control gain (7) given by

$$\hat{K} = Z\tilde{H}^{-1}, \quad (22)$$

is such that, for all initial conditions belonging to the estimated region of attraction $\mathcal{R}_{\mathcal{E}} = \mathcal{E}(P_i, \mu^{-1})$,

(1) for $\omega_k \neq 0$ with $\omega \in \mathcal{W}$, the trajectories of the closed-loop system do not leave the set $\mathcal{R}_{\mathcal{E}} = \mathcal{L}_{\mathcal{V}}(\mu^{-1}) \subseteq \mathcal{R}_{\mathcal{A}}$ for every initial state belonging to the set $\mathcal{R}_{\mathcal{E}0} = \mathcal{L}_{\mathcal{V}}(\beta^{-1}) \subseteq \mathcal{R}_0$, with $\beta^{-1} = \mu^{-1} - \delta^{-1}$, for all $k \geq 0$ and $\alpha_k \in \mathcal{P}$;
(2) $\|y\|_2^2 \leq \gamma(\|\omega\|_2^2 + V(z_0, \alpha_0))$, for $k \to \infty$;
(3) for $\omega_k = 0$, the set $\mathcal{R}_{\mathcal{E}} = \mathcal{L}_{\mathcal{V}}(\mu^{-1}) \subseteq \mathcal{R}_{\mathcal{A}}$ is a region of attraction for the system (9), for all $k \geq 0$.

Moreover, assuming $\mathcal{E}(P_i, \mu^{-1}) = \{z_k \in \mathbb{R}^{n+n_u} : z_k^\top P_i z_k \leq \mu^{-1}\}$, for $i = 1, \ldots, N$, an estimate of the region of attraction is given by

$$\mathcal{L}_{\mathcal{V}}(\mu^{-1}) = \bigcap_{\alpha_k \in \mathcal{P}} \mathcal{E}(P(\alpha_k), \mu^{-1}) = \bigcap_{i=1}^{N} \mathcal{E}(P_i, \mu^{-1}). \quad (23)$$

Proof. The ISS is investigated by considering the following parameter-dependent ISS-Lyapunov candidate function $V(\cdot,\cdot) : \mathbb{R}^{n+n_u} \times \mathcal{P} \to \mathbb{R}^+$:

$$V(z_k, \alpha_k) = z_k^\top P(\alpha_k) z_k, \quad P(\alpha_k) = \sum_{i=1}^{N} \alpha_{k(i)} P_i > \mathbf{0}. \quad (24)$$

If (24) fulfills the following conditions for $z_k \in \mathcal{R}_{\mathcal{E}} \subseteq \mathbb{R}^{n+n_u}$ and class $\mathcal{K}$ functions $\beta_1\|z_k\|^2$, $\beta_2\|z_k\|^2$, $\beta_3\|z_k\|^2$, and $\beta_4\|\omega\|^2$, $\beta_i > 0$, $i \in \{1, 2, 3, 4\}$:

$$\begin{aligned} \beta_1\|z_k\|^2 \leq V(z_k, \alpha_k) \leq \beta_2\|z_k\|^2, \\ \Delta V(z_k, \alpha_k) \leq -\beta_3\|z_k\|^2 + \beta_4\|\omega\|^2, \end{aligned} \quad (25)$$

for all allowed sequences of $\alpha_k \in \mathcal{P}$, then it ensures the regional stability of the closed-loop system, and the following level set can be associated:

$$\mathcal{L}_{\mathcal{V}}(\mu^{-1}) = \{z_k \in \mathbb{R}^{n+n_u} : V(z_k, \alpha_k) \leq \mu^{-1}, \forall \alpha_k \in \mathcal{P}\}, \quad (26)$$

with $0 < \mu < \infty$.

Admitting the feasibility of (16), then $L_{\mathtt{M}}$ and $L_{\mathtt{R}}$ are nonsingular. Also, from the positivity of $\tilde{P}_i$ and block $(2,2)$ of Θ_1, then $\tilde{H}$ is regular. Next, we do the replacements: $\tilde{P}_i = \tilde{H}^\top P_i \tilde{H}$, $\tilde{P}_j = \tilde{H}^\top P_j \tilde{H}$, $Z = \hat{K}\tilde{H}$, $L_{\mathtt{M}} = T_{\mathtt{M}}^{-1}$, $L_{\mathtt{R}} = T_{\mathtt{R}}^{-1}$, $X_{\mathtt{M}} = G_{\mathtt{M}}\tilde{H}$, $X_{\mathtt{R}_1} = G_{\mathtt{R}_1}\tilde{H}$, and $X_{\mathtt{R}_2} = G_{\mathtt{R}_2} L_{\mathtt{M}}$, multiply it by $\alpha_{k(i)}$, $\alpha_{k+1(j)}$, $\alpha_k \in \mathcal{P}$, and sum it for $i, j = 1, \ldots, N$, and pre- and post-multiply the resulting inequality by $\operatorname{diag}\{H, H, T_{\mathtt{M}}, T_{\mathtt{R}}, \mathbf{I}, \mathbf{I}\}$, where $H = \tilde{H}^{-\top}$, and its transpose (respectively). Then, we apply the Schur's complement and pre- and post-multiply it by $\xi = \begin{bmatrix} z_k^\top & z_{k+1}^\top & \psi_{\mathtt{M}}^\top & \psi_{\mathtt{R}}^\top & \omega_k^\top \end{bmatrix}^\top$ and its transpose, respectively. Considering $\Delta V(z_k, \alpha_k) = z_{k+1}^\top P(\alpha_{k+1}) z_{k+1} - z_k^\top P(\alpha_k) z_k$, and replacing $\hat{C}(\alpha_k) z_k$ for y_k, we get

$$\begin{aligned} &\Delta V(z_k, \alpha_k) - 2\psi_{\mathtt{M}k}^\top T_{\mathtt{M}} \psi_{\mathtt{M}k} \\ &\quad - 2\psi_{\mathtt{M}k}^\top T_{\mathtt{M}} G_{\mathtt{M}} z_k - 2\psi_{\mathtt{R}k}^\top T_{\mathtt{R}} \psi_{\mathtt{R}k} - 2\psi_{\mathtt{R}k}^\top T_{\mathtt{R}} G_{\mathtt{R}_1} z_k \\ &\quad\quad - 2\psi_{\mathtt{R}k}^\top T_{\mathtt{R}} G_{\mathtt{R}_2} \psi_{\mathtt{M}k} - \omega_k^\top \omega_k + \gamma^{-1} y_k^\top y_k < 0. \end{aligned} \quad (27)$$

Therefore, if (16) is verified and z_k belongs to both $\mathbb{S}_{\mathtt{M}}(\rho_{\mathtt{M}})$ and $\mathbb{S}_{\mathtt{R}}(\rho_{\mathtt{R}})$, i.e., conditions (14) are satisfied, then (27) yields $\beta_1 = \min \mathtt{eig}(\tilde{H}^{-\top} \tilde{P}_i \tilde{H}^{-1})$, $\beta_2 = \max \mathtt{eig}(\tilde{H}^{-\top} \tilde{P}_i \tilde{H}^{-1})$, with a small enough $\beta_3 > 0$, and $\beta_4 = 1$. Therefore, the inequalities (25) are verified and $V(z_k, \alpha_k)$ is an ISS-Lyapunov function, since $\Delta V(z_k, \alpha_k) \leq 0$ is checked. Also, the trajectories of the closed-loop system (9) under the control law (7) are bounded for any disturbance satisfying (2). Moreover, there is an upper bound for the ℓ_2-gain between the disturbance and the regulated output.

It remains to demonstrate that the generalized sector conditions (14)-(15) are in fact verified. The condition (17) ensures the inclusion of the contractive level set given by the Lyapunov function in $\mathbb{S}_{\mathtt{M}}(\rho_{\mathtt{M}})$. Assuming the feasibility of (17), we apply the change of variables $\tilde{P}_i = \tilde{H}^\top P_i \tilde{H}$, $Z = \hat{K}\tilde{H}$, and $X_{\mathtt{M}} = G_{\mathtt{M}}\tilde{H}$, multiply it by $\alpha_{k(i)}$ and sum it up for $i = 1, \ldots, N$. Next, we pre- and post-multiply it by $\operatorname{diag}\{\tilde{H}^{-\top}, 1\}$ and its transpose, respectively. Applying the Schur's complement and pre- and post-multiplying by $z_k^\top$ and z_k, if is verified that z_0 belongs to $\mathcal{L}_{\mathcal{V}}(\mu^{-1})$, it follows that $z_k^\top P(\alpha_k) z_k \leq V(z_0, \alpha_0) \leq \mu^{-1}$, leading to $\rho_{\mathtt{M}(r)}^{-2} |\Theta_{\mathtt{M}} z_k|^2 \leq \mu z_k^\top P(\alpha_k) z_k \leq \mu V(z_0, \alpha_0) \leq 1$. Thus, $|\Theta_{\mathtt{M}} z_k|^2 \leq \rho_{\mathtt{M}(r)}^2$, satisfying $\mathbb{S}_{\mathtt{M}}(\rho_{\mathtt{M}})$ given in (12).

Lastly, the feasibility of (18) ensures the inclusion of the contractive level set given by the Lyapunov function in $\mathbb{S}_{\mathtt{M}}(\rho_{\mathtt{M}}) \cap \mathbb{S}_{\mathtt{R}}(\rho_{\mathtt{R}})$, since (17) has already been verified. Then, replacing the following variables $\tilde{P}_i = \tilde{H}^\top P_i \tilde{H}$, $Z = \hat{K}\tilde{H}$, $X_{\mathtt{M}} = G_{\mathtt{M}}\tilde{H}$, $X_{\mathtt{R}_1} = G_{\mathtt{R}_1}\tilde{H}$, $X_{\mathtt{R}_2} = G_{\mathtt{R}_2} T_{\mathtt{M}}^{-1}$ and $L_{\mathtt{M}} = T_{\mathtt{M}}^{-1}$ in (18), multiplying it by $\alpha_{k(i)}$, and sum it up for $i = 1, \ldots, N$, pre- and post-multiplying by $\operatorname{diag}\{\tilde{H}^{-\top}, T_{\mathtt{M}}, 1\}$ and its transpose, respectively, applying the Schur's complement and pre- and post-multiplying by $\zeta^\top = [z_k^\top \;\; \psi_{\mathtt{M}k}^\top]$ and its transpose, respectively, it follows that $\Theta_{\mathtt{R}_1(r)} = \hat{\Lambda}_{(r)} + \Lambda\hat{K}_{(r)} - G_{\mathtt{R}_1(r)}$ and $\Theta_{\mathtt{R}_2(r)} = \Lambda_{(r)} - G_{\mathtt{R}_{2(r)}}$ for $r = 1, \ldots, n_u$, where $\Theta_{\mathtt{R}_1(r)} = \hat{\Lambda}_{(r)} + \Lambda\hat{K}_{(r)} - G_{\mathtt{R}_1(r)}$ and $\Theta_{\mathtt{R}_2(r)} = \Lambda_{(r)} - G_{\mathtt{R}_2(r)}$ for $r = 1, \ldots, n_u$. Similar to the analysis above mentioned for equation (17), it can be concluded that $|\Theta_{\mathtt{R}_1} z_k + \Theta_{\mathtt{R}_2} \psi_{\mathtt{M}k}|^2 \leq \rho_{\mathtt{R}(r)}^2$ if z_0 belongs to $\mathcal{R}_{\mathcal{E}} \equiv \mathcal{L}_{\mathcal{V}}(\mu^{-1})$, thus, z_0 belongs to both sets $\mathbb{S}_{\mathtt{M}}(\rho_{\mathtt{M}})$ and $\mathbb{S}_{\mathtt{R}}(\rho_{\mathtt{R}})$, given in (12) and (13), respectively, which ensures the generalized sector condition. Hence, the convergence to the origin of any trajectory of the closed-loop system (9) starting inside $\mathcal{L}_{\mathcal{V}}(\mu^{-1})$ for $\omega = \mathbf{0}$, is verified. □

Remark 6. The level set computation provided in (23) can be viewed as a particular version of (Figueiredo et al., 2021, Lemma 2, with $g = 1$) for Lyapunov functions affine on the parameters. The main advantage of (23) is that the level set are computed through finite dimensional conditions, providing the estimates of the region of attraction and the region of suitable initial conditions of system (9), called respectively by $\mathcal{R}_\mathcal{E} \subseteq \mathcal{R}_\mathcal{A}$ and $\mathcal{R}_{\mathcal{E}0} \subseteq \mathcal{R}_0$.

In case of no external disturbance, i.e., $\omega = \mathbf{0}$ for all $k \geq 0$, the following corollary can be stated as a special case of Theorem 5.

Corollary 7. Suppose that there exist symmetric and positive definite matrices $\tilde{P}_i \in \mathbb{R}^{(n+n_u)\times(n+n_u)}$, diagonal matrices $L_\mathtt{M}$, $L_\mathtt{R} \in \mathbb{R}^{n_u \times n_u}$, matrices Z, $X_\mathtt{M}$, $X_{\mathtt{R}_1} \in \mathbb{R}^{n_u\times(n+n_u)}$, $X_{\mathtt{R}_2} \in \mathbb{R}^{n_u \times n_u}$, $\tilde{H} \in \mathbb{R}^{(n+n_u)\times(n+n_u)}$, for i, $j = 1, \dots, N$, and a positive scalar μ such that, $\Theta_1 < \mathbf{0}$, (17), and (18) are feasible for $\forall r = 1, \dots, n_u$. Then, the control gain matrices given by (22) yield the control law (7), ensuring the closed-loop asymptotic stability for all initial conditions belonging to $\mathcal{R}_\mathcal{E} = \mathcal{L}_\mathcal{V}(\mu^{-1}) \subseteq \mathcal{R}_\mathcal{A}$.

3.1 Optimization procedures

The conditions for controller synthesis and estimation of the region of attraction can be explored in different objectives through convex optimization procedures as follows.

Maximization of the disturbance tolerance: The procedure's objective is to determine the robust control gain maximizing the set of admissible perturbations $\omega \in \mathcal{W}$ for a set of allowable initial states $\mathcal{R}_0$. An interesting particular case can be investigated when the system is in equilibrium, i.e., $z_0 = \mathbf{0}$. In such a case, we have $\delta^{-1} = \mu^{-1}$, which allows us to formulate the next convex optimization procedure:

$$\Pi_1 : \begin{cases} \displaystyle\min_{\substack{\tilde{P}_{i,j}, L_\mathtt{M}, L_\mathtt{R}, Z, \gamma \\ X_\mathtt{M}, X_{\mathtt{R}_1}, X_{\mathtt{R}_2}, \tilde{H}}} \mu \\ \text{s.t. LMIs (16)–(19)}, \forall(i,j) = \{1, \dots, N\}^2. \end{cases} \quad (28)$$

Minimization of the ℓ_2-gain: The objective is designing the robust control gain that minimizes the ℓ_2-gain between the disturbance signal α_k and the output y_k, for a given disturbance energy limit, δ^{-1}. Thus, the following convex optimization procedure minimizes the ℓ_2-gain:

$$\Pi_2 : \begin{cases} \displaystyle\min_{\substack{\tilde{P}_{i,j}, L_\mathtt{M}, L_\mathtt{R}, Z, \mu \\ X_\mathtt{M}, X_{\mathtt{R}_1}, X_{\mathtt{R}_2}, \tilde{H}}} \gamma \\ \text{s.t. LMIs (16)–(19)}, \forall(i,j) = \{1, \dots, N\}^2. \end{cases} \quad (29)$$

4. NUMERICAL EXAMPLES

4.1 Example 1:

Consider system (1) adapted from (de Souza et al., 2019) as $A = 2(1+\theta)$, $B = 1(1+\theta)$, $B_w = 0.1(1+\theta)$, $C = 0.1(1+\theta)$, where the uncertain parameter $|\theta| \leq 0.1$, the actuator parameter $\Lambda = 15$, and the symmetric limits of magnitude and rate saturation given by $\rho_\mathtt{M} = 0.7$ and $\rho_\mathtt{R} = 0.3$.

We used the optimization procedure Π_1 in (28) to design a robust feedback gain such that, assuming a null initial condition, i.e., $z_k = \mathbf{0}$, the set of admissible disturbances is maximized. Thus, we have $\delta^{-1} = \mu^{-1}$. Consequently, we got $\mu = 1.6602$, i.e., $\|\omega\|_2^2 \leq \delta^{-1} = \mu^{-1} = 0.6023$, and the robust gain vector $\hat{K} = [-0.1275 \ 0.8738]$.

Therefore, using the robust gain obtained, the closed-loop responses were simulated for the application of an energy disturbance vector with the form $\omega = [0.7761 \ \mathbf{0}]$, i.e., with the energy bound obtained by the optimization procedure with different sequences of the varying parameters α_k. In Figure 2, the state trajectories are presented by the blue lines; observe that all of them converge to the origin and do not leave the $\mathcal{R}_\mathcal{E}$ region, defined by the intersection of the ellipsoids $\mathcal{E}(P_1, \mu^{-1})$ and $\mathcal{E}(P_2, \mu^{-1})$ (solid and dashed black lines, respectively), given by the matrices of each vertex of the Lyapunov function

$$P_1 = \begin{bmatrix} 53.4607 & 39.0182 \\ 39.0182 & 30.8344 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 56.2748 & 41.1667 \\ 41.1667 & 32.3278 \end{bmatrix}.$$

Notably, the trajectories approach the limiting edge of the region of attraction. When applying a perturbation vector with energy 139% greater than the energy bound obtained by the optimization procedure, i.e., $\omega_k = [1.8549 \ \mathbf{0}]$, with different sequences of the varying parameters α_k, yields the trajectories presented by the magenta lines. Note that one of the trajectories presents an unstable behavior, leaving the region of attraction. Although some sequences of α_k may lead to (regional) stable behavior, our conditions cannot guarantee ISS for all possible sequences of α_k under a disturbance with energy superior to δ^{-1}.

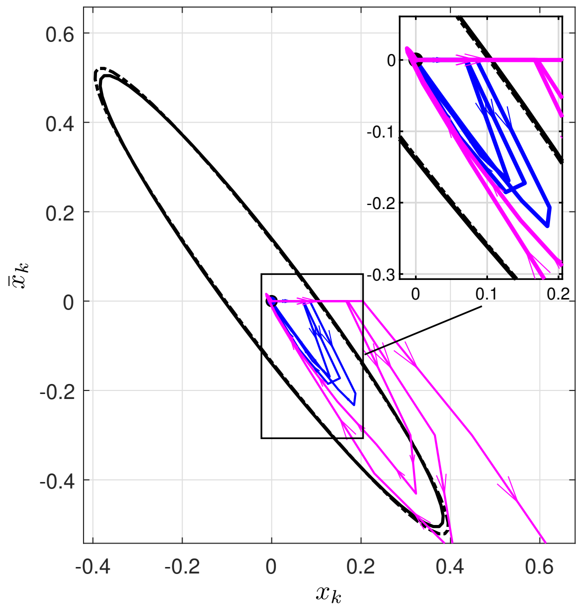

Figure 2. Trajectories of the system subject to the maximum admissible disturbance energy (blue lines) limited by $\mathcal{R}_\mathcal{E}$ and to a disturbance whose energy exceeds the disturbance limit (magenta lines), for different sequences of the varying parameters.

In the top plot of Figure 3, we see the plant state, and the bottom plot shows the actuator's output during the simulation for the stable cases respecting the disturbance limit determined by the optimization procedure and for different sequences of α_k. It is noticed that the applied disturbance leads the actuator near to the limit of the variation rate, but not reaching it, and the system stays stable in the presence of the exogenous signal.

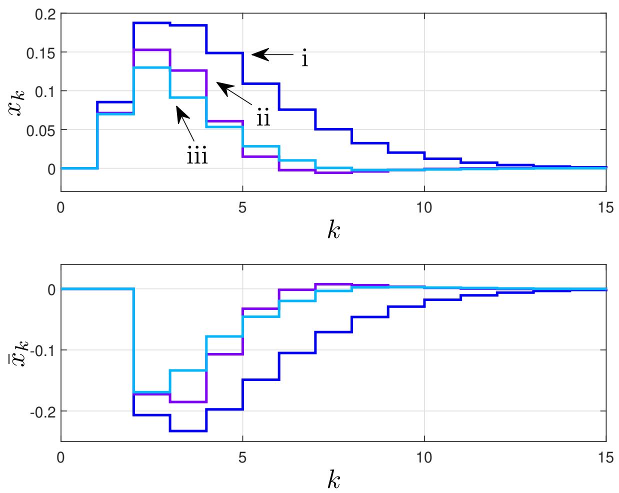

Figure 3. Simulations of the plant states (top) and actuator's outputs (bottom) in the presence of the maximum admissible disturbance for distinct sequences of the varying parameters, such that i. $\alpha_{k(1)} = 0.01k$; ii. $\alpha_{k(1)} = \text{abs}(\sin(1 + k))$; iii. $\alpha_{k(1)} = \text{abs}(\sin(2 + k/0.01))$. In all cases, $\alpha_{k(2)} = 1 - \alpha_{k(1)}$.

4.2 Example 2

Consider the precisely known discrete-time pendulum model given by (1) investigated by Gomes da Silva Jr. et al. (2007), with matrices

$$A = \begin{bmatrix} 1.0013 & -0.0500 & -0.0013 \\ -0.0500 & 1.0025 & 0.0500 \\ -0.0013 & 0.0050 & 1.0013 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$B = B_w = [-0.0021 \; 0.1251 \; 5.0021]^\top \times 10^{-2}$, with the actuator parameter $\Lambda = 20$, the magnitude and rate limits $\rho_{\mathtt{M}} = 1.25$ and $\rho_{\mathtt{R}} = 2$, respectively. Assuming a null initial condition, i.e., $z_k = \mathbf{0}$, we used the optimization procedure Π_2 given in (29) in order to design robust controllers that minimizes the ℓ_2-gain for different values of maximum admitted perturbation energy (δ^{-1}), and compared the results with those obtained in (Gomes da Silva Jr. et al., 2007). For $\delta^{-1} = \{1, 2, 4, 8\}$ we got the ℓ_2-gains as $\gamma = \{2.8284, 4.5576, 10.1845, 57.7117\}$. For the same disturbance limits, (Gomes da Silva Jr. et al., 2007) got $\gamma = \{2.2508, 4.7053, 14.5536, 211.8103\}$. As we can see, unless for $\delta^{-1} = 1$, our approach yields better results, resulting in control gains that better mitigates high disturbance effects.

In cases where the pendulum model may present uncertainties, consider that $A = A + F\theta W$, with $F = [1 \; 0 \; 1]$, $W = [0.05 \; 0 \; 0.01]^\top$, and $|\theta| \leq 0.001$. For $\delta^{-1} = \{1, 2, 4, 8\}$ for the same maximum disturbance limits used before, the optimization procedure Π_2 yields to the ℓ_2-gains $\gamma = \{2.8471, 4.5927, 10.3391, 62.036\}$. Predictably, the values of the ℓ_2-gains slightly increased, resulting in less effective disturbance attenuation than the precisely known system. However, see that the Theorem 5 can be used in uncertain systems, while the method in (Gomes da Silva Jr. et al., 2007) does not apply for such a case.

5. CONCLUSIONS

We have presented new LMI conditions that efficiently solve the robust controller design that ensures the regional input-to-state stability of time-varying discrete-time systems under magnitude and rate saturating actuators. By handling exogenous energy signals, the feasibility of the proposed conditions allows computing an estimate of the region of attraction, $\mathcal{R}_\mathcal{E}$. Moreover, it allows handling some practical control issues, including minimizing the ℓ_2-gain between the measured output and the disturbance signal or maximizing the tolerable disturbance energy. Our approach has been compared with other results from the literature with the aid of two numerical examples. The results suggest better performance of our proposal.

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