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An Application of QSR-Dissipativity to the Problem of Static Output Feedback Robust Stabilization of Nonlinear Systems

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Abstract: In this work we deal with the asymptotic stabilization problem of polynomial (and rational) input-affine systems subject to parametric uncertainties. The problem of linear static output feedback (SOF) control synthesis is handled, having as a prerequisite a differential algebraic representation (DAR) of the plant. Using the property of strict QSR-dissipativity, the Finsler's Lemma and the notion of linear annihilators we introduce a new dissipativity-based strategy for robust stabilization which determines a static feedback gain by solving a simple linear semidefinite program on a polytope. At the same time, an estimate of the closed-loop domain of attraction is given in terms of an ellipsoidal set. The novelty of the proposed approach consists in this combination of dissipativity theory and powerful semidefinite programming (SDP) tools allowing for a simple solution of the challenging problem of static output feedback design for nonlinear systems. A numerical example allows the reader to verify the applicability of the proposed technique.

Resumo: O presente artigo trata do problema de estabilização assintótica de sistemas polinomiais (e racionais) que são afins no controle e sujeitos a incertezas paramétricas. O problema de controle via realimentação linear e estática de saída é abordado, tendo como premissa uma representação algébrico-diferencial da planta. Utilizando a propriedade da QSR-dissipatividade, o Lema de Finsler e a noção de aniquilador linear, apresentamos uma nova estratégia de estabilização robusta baseada em dissipatividade e que determina um ganho estático de realimentação por meio da resolução de um problema de programação semidefinida simples em um politopo. Ao mesmo tempo, uma estimativa do domínio de atração em malha fechada é dada em termos de um elipsóide. A contribuição da estratégia proposta consiste na combinação entre teoria da dissipatividade e poderosas ferramentas de programação semidefinida, resultando em uma solução relativamente simples para um problema de controle desafiador que é o problema da realimentação estática de saída de sistemas não lineares.

Keywords: Static Output Feedback, Robust Control, QSR-Dissipativity, Differential Algebraic Representation.

Palavras-chave: Realimentação Estática de Saída, Controle Robusto, QSR-Dissipatividade, Representação Algébrico-Diferencial.

1. INTRODUCTION

Designing a static output feedback (SOF) controller is widely regarded as a challenging stabilization problem. From a theoretical point of view, even in the case of the linear and time-invariant (LTI) systems the question of whether there exists a simple and testable (necessary and sufficient) condition for stabilizability remains unanswered. In spite of the number of controller design techniques proposed over the last decades a definite solution to this problem is yet to appear. See Veselý (2001), Crusius and Trofino (1999), Apkarian and Noll (2006), Gahinet and Apkarian (2011), and Sadabadi and Peaucelle (2016) for a comprehensive overview of the subject. Furthermore, when dealing with the more complex case of the nonlinear plants, linear SOF design becomes even more difficult, as one can not as readily apply semidefinite programming (SDP) tools in this scenario, as opposed to the numerous sufficient conditions based on linear matrix inequalities (LMIs) available in the LTI context (Sadabadi and Peaucelle (2016)).

For many reasons, SOF design remains a relevant research topic from both a theoretical and a practical point of view. Firstly, it is not always possible to measure every single state component in order to implement a full state feedback control law. In practice, and quite frequently, one is given access to only partial state information. Moreover, a static gain is a very simple controller which is also capable of dealing with robust stabilization issues in practical implementations.

In the present work, we solve the linear SOF robust control problem by means of a new dissipativity-based strategy. The concept of dissipativity was introduced a few decades ago and has ever since proved profoundly fruitful for stability analysis and controller design (Willems (1972), Brogliato et al (2020)). It applies for general input-affine systems, let them be square or not, open-loop stable or unstable (Khalil (2002)). Under certain conditions, dissipative systems are Lyapunov stable, asymptotically and even exponentially stable, or they can be proved to be stabilizable by some suitable feedback control law (Brogliato et al (2020)). Dissipativity is a generalization of the notion of passivity and both properties have been applied for the sake of feedback stabilization of numerous classes of systems (Hill and Moylan (1976), Astolfi et al (2002), Feng et al (2013)). Passivity- and dissipativity-based control comprise quite a broad and mature research field, where many interesting applications have been reported (Ortega and García-Canseco (2004), Shishkin and Hill (1995)).

In Madeira (2018), a special case of dissipativity called (strict) QSR-dissipativity was shown to be necessary and sufficient for the linear SOF stabilizability of LTI systems, under certain circumstances. A linear SDP strategy for controller design was proposed as well and in that same reference QSR-dissipativity was also applied for SOF stabilization of rational nonlinear systems without uncertainties. In the nonlinear case, though, dissipativity was proved only sufficient for SOF asymptotic stabilization. By making use of the well-known Finsler's Lemma, the notion of linear annihilators and by adopting a polytopic approach the problem of linear SOF asymptotic stabilization of that class of systems was solved locally. It was shown that under a few assumptions QSR-dissipativity can ensure closed-loop asymptotic stability in a domain, whereas a differential algebraic representation (DAR) of the plant was used in order to allow for a linear SDP formulation of the control problem. DAR representations and their application for open-loop robust stability analysis were extensively investigated in Trofino and Dezuo (2014). Although the problem of controller design was not addressed in that reference, the main foundations of a polytopic approach of rational systems based on the Finsler's Lema and linear annihilators are due to it and to Coutinho et al (2002), and Coutinho et al (2008). Applications of this technique followed in Polcz et al (2015), Madeira and Adamy (2016) and in Madeira (2018), for instance.

In Azizi (2017) and Azizi et al (2018), robust stability analysis and state feedback control synthesis for rational nonlinear systems with a DAR and uncertainties were studied, and the case of the dynamical systems with saturation in the input was also considered. Nonetheless, linear SOF control was not addressed, nor the notion of dissipativity was applied. Although it was suggested in Azizi (2017) the application of the notion passivity as a possible future research direction, we believe that it is actually the concept of dissipativity which provides a more fruitful approach, as it allows for a simple linear SDP formulation of the problem of controller design and it also applies for general nonsquare systems.

In this paper, we extend the dissipativity-based results of Madeira (2018) for the context of robust stabilization. At the same time that a controller is designed, a domain of attraction for the closed-loop system is estimated as an ellipsoidal set. See Polcz et al (2015), Valmorbida and Anderson (2014), and Chesi (2004) for further references on the subject of domain of attraction estimation. Furthermore, our results trivially apply for linear state feedback control as well, as one only has to consider the system output as equal to the state variable, i.e. $y = x$. Thus, state feedback and SOF can be handled by the same stabilization framework based on the notion of dissipativity.

The content of this paper is organized as follows. In Section 2 we have some preliminary results relevant to this work. Then in Section 3 our main results are presented, i.e. a new dissipativity-based strategy for linear SOF robust stabilization of polynomial and rational systems, and a simple condition for estimating the closed-loop domain of attraction. Section 4 contains a numerical example and Section 5 provides the concluding remarks of this paper.

2. PRELIMINARIES

2.1 Nonlinear Systems and DARs

Consider an input-affine and uncertain nonlinear system as given in Azizi (2017)

$$\dot{x}(t) = f(x(t), \delta(t)) + g(x(t), \delta(t))u(t), \tag{1}$$

$t \geq 0$, $x(0) = x_0$, where in this paper we restrict the system's output to be linear

$$y(t) = h(x(t)) = Cx(t). \tag{2}$$

Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a compact set, with $0 \in \mathcal{X}$, such that $x(t) \in \mathcal{X}$ is a state vector of this dynamical system. $\delta(t) \in \mathcal{D} \subset \mathbb{R}^l$ is an uncertain and bounded parameter vector which accounts for deviations of the model description around its nominal part. Functions $f : \mathcal{X} \times \mathcal{D} \to \mathbb{R}^n$ and $g : \mathcal{X} \times \mathcal{D} \to \mathbb{R}^{n\times m}$ are such that $(f, g) \in C^1$, $f(0, \delta) = 0$ for all $\delta \in \mathcal{D}$, and the origin $(x(t), u(t)) \equiv (0, 0)$ is an equilibrium point of (1). Furthermore, The control signal $u(t)$ is a measurable function, with $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ for all $t \geq 0$, $0 \in \mathcal{U}$ (Haddad and Chellaboina (2008)). Finally, $h : \mathcal{X} \to \mathbb{R}^p$ with $C \in \mathbb{R}^{p\times n}$ a constant matrix, and functions (f, g) are polynomial or at most rational on their arguments.

A dynamical system (1)-(2) can be represented in many different and equivalent ways. In the case of polynomial or rational models, a much convenient representation is the well-known Differential Algebraic Representation (DAR), which is referred to as providing less conservative results than Linear Fractional Representations (LFR) and Linear Parameter Varying (LPV) forms. A DAR is more general than an LFR and an LPV approach, and it usually leads two larger estimates of a domain of attraction (Azizi (2017)). A DAR of an input-affine plant (1)-(2) is given by

$$\dot{x}(t) = A_1(x, \delta)x(t) + A_2(x, \delta)\pi(t) + A_3(x, \delta)u(t), \tag{3}$$

$$0 = \Pi_1(x, \delta)x(t) + \Pi_2(x, \delta)\pi(t) + \Pi_3(x, \delta)u(t), \tag{4}$$

$$y(t) = Cx(t). \tag{5}$$

Here, $\pi(x, u, \delta) \in \mathbb{R}^{n_\pi}$ is a suitably chosen vector of nonlinear functions. $A_1(x, \delta) \in \mathbb{R}^{n\times n}$, $A_2(x, \delta) \in \mathbb{R}^{n\times n_\pi}$,

$A_3(x,\delta) \in \mathbb{R}^{n \times m}$, $\Pi_1(x,\delta) \in \mathbb{R}^{n_\pi \times n}$, $\Pi_2(x,\delta) \in \mathbb{R}^{n_\pi \times n_\pi}$, $\Pi_3(x,\delta) \in \mathbb{R}^{n_\pi \times m}$ are matrices of affine functions with respect to (x,δ), where $\Pi_2(x,\delta)$ is a square full-rank matrix for all vectors $(x,\delta) \in \mathcal{X} \times \mathcal{D}$.

The DAR of a system is not unique and the a state-space representation (1)-(2) is well-posed in its DAR form if $\Pi_2(x,\delta)$ is invertible, as from (4) and (3) we have

$$\pi(x,u,\delta) = \Pi_2^{-1}(x,\delta)[-\Pi_1(x,\delta)x - \Pi_3(x,\delta)u], \quad (6)$$

$$\dot{x}(t) = (A_1 - A_2\Pi_2^{-1}\Pi_1)x(t) + (A_3 - A_2\Pi_2^{-1}\Pi_3)u(t). \quad (7)$$

Furthermore, as we are dealing with input-affine systems, the following relation can actually replace (4)

$$0 = \Pi_d(x,\delta)x_d + \Pi_2(x,\delta)\pi, \quad (8)$$

with

$$x_d^\top = \begin{bmatrix} x^\top & u^\top \end{bmatrix}, \quad (9)$$

$$\Pi_d = [\Pi_1 \ \Pi_3], \quad (10)$$

where $\Pi_d(x,\delta) \in \mathbb{R}^{n_\pi \times n_d}$ and $n_d = n+m$. From this point on, we always refer to DARs of type (3)-(8)-(5) subject to (9), with $x_d \in \mathbb{R}^{n_d}$.

2.2 Finsler's Lemma and Linear Annihilators

From Trofino and Dezuo (2014), the following version of the Finsler's Lemma is presented.

Lemma 1. Consider $\mathcal{W} \subseteq \mathbb{R}^{n_s}$ a given polytopic set, and let $Q_d : \mathcal{W} \to \mathbb{R}^{n_q \times n_q}$ and $C_d : \mathcal{W} \to \mathbb{R}^{n_r \times n_q}$ be given matrix functions, with Q_d symmetric. Then, the following statements are equivalent

(i) $\forall w \in \mathcal{W}$ the condition that $z^\top Q_d(w) z > 0$ is satisfied $\forall z \in \mathbb{R}^{n_q}$: $C_d(w)z = 0$.

(ii) $\forall w \in \mathcal{W}$ there exists a certain matrix function $L : \mathcal{W} \to \mathbb{R}^{n_q \times n_r}$ such that $Q_d(w) + L(w)C_d(w) + C_d(w)^\top L^\top(w) > 0$.

If Q_d and C_d are affine functions of w and L is a constant matrix to be determined, then (ii) becomes a polytopic LMI condition which is sufficient for (i). Here, in accordance with the notation introduced in the previous section, we consider

$$w^\top = \begin{bmatrix} x_d^\top & \delta^\top \end{bmatrix} = \begin{bmatrix} x^\top & u^\top & \delta^\top \end{bmatrix}, \quad (11)$$

$$\mathcal{X}_d = \mathcal{X} \times \mathcal{U}, \quad \mathcal{W} = \mathcal{X}_d \times \mathcal{D}, \quad (12)$$

$$n_s = n_d + l = n + m + l, \quad n_r = n_\pi, \quad (13)$$

$$n_q = n_d + n_\pi = n + m + n_\pi. \quad (14)$$

Also from Trofino and Dezuo (2014), the following definition plays a key role in the forthcoming sections.

Definition 1. Given a function $\hat{l} : \mathbb{R}^{n_q} \to \mathbb{R}^{n_v}$ and a positive integer n_r, a matrix function $\mathcal{N}_{\hat{l}}$: $\mathbb{R}^{n_q} \to \mathbb{R}^{n_r \times n_v}$ is called an *annihilator* of $\hat{l}$ if

$$\mathcal{N}_{\hat{l}}(z)\, \hat{l}(z) = 0, \quad (15)$$

$\forall z \in \mathbb{R}^{n_q}$ of interest. If in addition $\mathcal{N}_{\hat{l}}$ is a linear function, then it is said to be a *linear annihilator*.

Suppose for instance that $\hat{l}(z) = z = [z_1 \ldots z_{n_q}]^\top \in \mathbb{R}^{n_q}$. Then, considering all possible pairs (z_i,z_j) for $i \neq j$ without repetition, i.e. $\forall(i,j)$ $(j > i)$, a general closed-form expression for a linear annihilator is as follows

$$\mathcal{N}_{\hat{l}}(z) = \begin{bmatrix} \Phi_1(z) & Y_1(z) \\ \vdots & \vdots \\ \Phi_{(n_q-1)}(z) & Y_{(n_q-1)}(z) \end{bmatrix}, \quad (16)$$

where

$$\begin{aligned} Y_i(z) &= -z_i I_{(n_q-i)}, \ \ i \in \mathbb{I}_{(n_q-1)}, \\ \Phi_1(z) &= [z_2 \ \ldots \ z_{n_q}]^\top, \\ \Phi_i(z) &= \begin{bmatrix} & z_{(i+1)} \\ 0_{(n_q-i)\times(i-1)} & \vdots \\ & z_{n_q} \end{bmatrix}, \ \ i \in \{2,\ldots,n_q-1\}, \end{aligned} \quad (17)$$

$\mathcal{N}_{\hat{l}}(z) \in \mathbb{R}^{n_r \times n_q}$ with $n_r = \sum_{j=1}^{n_q-1} j$.

Linear annihilators are not unique, which means that (16)-(17) provide only one among multiple solutions to the problem. Furthermore, notice that $\hat{l}(z) = z$ is a very simple (linear) vector to which a linear annihilator can be easily found. From (8)-(9), though, we will have to determine in this work a suitable $\mathcal{N}_{\hat{l}}$ for a vector

$$\hat{l}(w) = \hat{l}(x_d,\delta) = \begin{bmatrix} x_d \\ \pi \end{bmatrix}, \quad (18)$$

which contains both linear and nonlinear (polynomial and rational) functions of x and δ. For further details on this topic and a systematic procedure for determining linear annihilators the reader is referred to Trofino and Dezuo (2014) and Coutinho et al (2008).

2.3 Dissipativity

Firstly, consider a state-space model (1)-(2) with $\delta = 0$

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad (19)$$

$$y(t) = h(x(t)) = Cx(t). \quad (20)$$

The definition of dissipativity applies for systems (19)-(20) that are completely reachable. It demands the existence of a locally integrable *supply rate* $r(u(t),y(t))$ and a so-called *storage function* $V(x)$ for the system, $V : \mathcal{X} \to \mathbb{R}$, $V \in C^1$ (Haddad and Chellaboina (2008)). From Brogliato et al (2020), we present below a few definitions which will soon be applied for feedback stabilization of nonlinear systems.

Definition 2. A system is said to be *dissipative* if there exists a storage function $V(x) \geq 0$ such that the following dissipation inequality holds

$$\dot{V}(x) \leq r(u,y), \quad (21)$$

along all possible trajectories of (19)-(20) starting at $x(0)$, for all $x(0)$, $t \geq 0$.

Definition 3. A dynamical system is called *QSR-dissipative* if it is dissipative with the following supply rate

$$r(u,y) = y^\top Q y + 2y^\top S u + u^\top R u, \quad (22)$$

where Q and R are symmetric.

Matrices $Q \in \mathbb{R}^{p\times p}$, $S \in \mathbb{R}^{p\times m}$ and $R \in \mathbb{R}^{m\times m}$ are real and appear linearly in (22). A relevant result known in literature is that if $Q \leq 0$ and $V(x) > 0$, then the origin $(x(t) \equiv 0)$ of the free system $(u(t) \equiv 0)$ (19) is stable in the sense of Lyapunov.

Definition 4. A system is said to be *strictly QSR-dissipative* if it is QSR-dissipative and there exists $T(x) > 0$ such that

$$\dot{V} + T \leq y^\top Q y + 2y^\top S u + u^\top R u, \quad (23)$$

where Q and R are symmetric.

If a system is strictly QSR-dissipative with $Q \leq 0$ and $V(x) > 0$, then the free system is asymptotically stable (Haddad and Chellaboina (2008)). For a system (19)-(20) without uncertainties the following dissipativity condition is equivalent to (23)

$$t(x,u) = -\nabla V(x)^\top [f(x) + g(x)u] - T(x) + h(x)^\top Q h(x) + 2h(x)^\top S u + u^\top R u \geq 0, \quad (24)$$

where (24) is clearly a function of the augmented variable x_d defined in (9). In this context, system (19)-(20) is said to be *locally strictly QSR-dissipative* (Pota and Moylan (1993)) if $t(x,u) \geq 0$ in some domain $(x,u) \in \mathcal{X} \times \mathcal{U}$ containing the point $(x(t), u(t)) \equiv (0,0)$. Nevertheless, in this work we consider uncertain systems (1)-(2) which are supposed to be *robust locally strictly QSR-dissipative*, i.e.

$$t(x_d,\delta) = -\nabla V(x)^\top [f(x,\delta) + g(x,\delta)u] - T(x) + h(x)^\top Q h(x) + 2h(x)^\top S u + u^\top R u \geq 0, \quad (25)$$

for all $(x_d, \delta) \in \mathcal{X}_d \times \mathcal{D}$.

In this paper, we consider quadratic Lyapunov functions $V(x)$ which are independent of the uncertainty δ,

$$V = x^\top P x, \quad P > 0, \quad (26)$$

such that the following ellipsoidal set can be defined (Rohr et al (2009))

$$\mathcal{E}(P,1) = \left\{ x \in \mathbb{R}^n; x^\top P x \leq 1 \right\}, \quad (27)$$

which will be useful for estimating a domain of attraction for closed-loop asymptotic stability. In the forthcoming sections we also restrict the function $T(x)$ to be quadratic

$$T = x^\top N x, \quad N > 0. \quad (28)$$

3. STATIC OUTPUT FEEDBACK DESIGN

Our next step consists in connecting Lemma 1 and condition (25) in order to design an asymptotically stabilizing controller for nonlinear uncertain plant (3)-(8)-(5). In this section, $\mathcal{X}$ and $\mathcal{D}$ are considered to be polytopic sets. From Rohr et al (2009), a polytope $\mathcal{X}$ with n_x vertices can be represented as the intersection of n_{xe} hyperplanes

$$\mathcal{X} = \left\{ x \mid a_k^\top x \leq 1, k = 1, \cdots, n_{xe} \right\}, \quad (29)$$

where the constant vectors $a_k \in \mathbb{R}^n$ can be determined by fulfilling $a_k^\top x = 1$ at all groups of adjacent vertices of $\mathcal{X}$. A similar description for the set $\mathcal{D}$ is also possible through $n_{\delta e}$ hyperplanes.

Then recover that we apply Lemma 1 subject to (11)-(14), where x_d represents the augmented variable defined in (9) and suppose, in addition, that (8) holds subject to

(1) $\pi : \mathcal{X}_d \times \mathcal{D} \to \mathbb{R}^{n_\pi}$ is a vector of nonlinear functions. As (1) is affine in the input, then $\pi(x_d, \delta)$ must be either affine in u or independent of it, i.e. $\pi(x, \delta)$.
(2) $\Pi_d : \mathcal{X} \times \mathcal{D} \to \mathbb{R}^{n_\pi \times n_d}$ and $\Pi_2 : \mathcal{X} \times \mathcal{D} \to \mathbb{R}^{n_\pi \times n_\pi}$ are affine matrix functions of (x, δ).
(3) matrix $\Pi_2(x,\delta)$ is invertible for all values of $(x,\delta) \in \mathcal{X} \times \mathcal{D}$.

Next, suppose that $t(x_d, \delta)$ can be decomposed in the following manner

$$t(x_d, \delta) = \pi_d^\top Q_d \pi_d, \quad (30)$$

$$\pi_d = \begin{bmatrix} x_d \\ \pi \end{bmatrix}, \quad (31)$$

where Q_d is symmetric, affine on (x, δ), and linear on all the unknown coefficients of (Q, S, R, P, N) for (x, δ) fixed. The vector $\pi(x_d, \delta)$ is a basis from which we can represent *both* (1)-(2) in its DAR form (3)-(8)-(5) and the dissipativity condition $t(x_d, \delta)$ as in (30). Notice that matrix function $Q_d(x, \delta)$ contains all variables one has to determine for guaranteeing the robust local dissipativity of (1)-(2).

The nonlinear basis π is not a function of the real coefficients of (Q, S, R, P, N), and for a systematic procedure to determine this vector see Trofino and Dezuo (2014). In $t(x_d, \delta)$, all nonlinear and rational terms are constrained to the vector π. The terms of $Q_d(x, \delta)$ are polynomial and consist in simple multiplications of the coefficients of (Q, S, R, V, T) and the components of (x, δ). In addition, consider

$$C_d(x, \delta) = \left[\Pi_d(x, \delta) \ \Pi_2(x, \delta) \right], \quad (32)$$

as a linear annihilator of π_d. This function provides an extra degree of freedom when investigating the feasibility of $t(x_d, \delta) \geq 0$ in a domain.

Remark 1. It is important to stress that Q_d is not a function of u, although condition (30) also depends on the input. This is the case because $t(x_d, \delta)$ is quadratic in u, such that it is only the vector function π_d which depends on the control signal.

In the sequel we establish our main result, a theorem which connects well-established concepts in the field of polytopic LMI conditions to the SOF control problem. The theorem provides a constructive approach for robust local dissipativity analysis and controller design for polynomial and rational uncertain systems.

Theorem 1. Let us consider a polynomial or rational uncertain dynamical system (3)-(8)-(5) with an equilibrium at $(x, u) = (0, 0)$. Furthermore, let $(x, \delta) \in \mathcal{X} \times \mathcal{D}$ be a given polytope around $(x, \delta) = (0, 0)$, with $\mathcal{X}$ described by (29). For a set of real matrices (Q, S, R) and positive functions V and T given by (26) and (28), with real P and N, assume that a suitable decomposition (30)-(31) of dissipativity condition (25) is given. In addition, with a linear annihilator C_d given in (32), suppose that the following LMI is fulfilled for all (x, δ) at the vertices of $\mathcal{X} \times \mathcal{D}$

$$Q_d + L_d C_d + C_d^\top L_d^\top > 0, \quad (33)$$

for some constant matrix L_d, and for some set of real coefficients of (Q, S, R, P, N) subject to

$$P > 0, \quad N > 0, \quad R > 0. \quad (34)$$

If $\Delta \geq 0$, where

$$\Delta = S R^{-1} S^\top - Q, \quad (35)$$

and at the same time

$$\begin{bmatrix} P & a_k \\ a_k^\top & 1 \end{bmatrix} \geq 0, \text{ for all } k = 1, \cdots, n_{xe}, \quad (36)$$

then the following linear SOF

$$u = Ky, \quad K = -R^{-1} S^\top, \quad (37)$$

asymptotically stabilizes the system around the origin, and the ellipsoid $\mathcal{E}(P, 1) \subset \mathcal{X}$ is an estimate of the closed-loop domain of attraction.

Proof: Firstly, for functions V and T of fixed degrees (for example, quadratic functions), dissipativity condition (25) can always be decomposed as in (30)-(31), whereas a linear annihilator C_d is given in (32). From Lemma 1, $t(x_d, \delta) > 0$ for all $(x_d, \delta) \in \mathcal{X}_d \times \mathcal{D}, \mathcal{U} = \mathbb{R}^m$, if condition (33) is fulfilled at all vertices of the polytope $\mathcal{X} \times \mathcal{D}$ for some constant matrix L_d and for a set of coefficients of (Q, S, R, P, N). Notice that (33) is independent of u, as $t(x_d, \delta)$ is quadratic in the input. Furthermore, since $N > 0$, the system is robust locally strictly QSR-dissipative in $(x_d, \delta) \in \mathcal{X}_d \times \mathcal{D}$.

Under (37) and $R > 0$, $t(x_d, \delta) > 0$ means that

$$\nabla V^\top [f - gR^{-1}S^\top h] < -T - h^\top \Delta h. \tag{38}$$

If $\Delta \geq 0$, then asymptotic stability is guaranteed for all $(x, \delta) \in \mathcal{X} \times \mathcal{D}$, as

$$\nabla V^\top [f - gR^{-1}S^\top Cx] = \nabla V^\top [f + gu] < 0. \tag{39}$$

From (36),

$$P - a_k a_k^\top \geq 0, \tag{40}$$

for all $k = 1, \cdots, n_{xe}$, and

$$(x^\top a_k)(a_k^\top x) \leq x^\top P x, \tag{41}$$

such that if $x \in \mathcal{E}(P, 1)$ then $x \in \mathcal{X}$, with $\dot{V} < 0$ inside the whole polytope. Then, for all $x(0) \in \mathcal{E}(P, 1)$, the state trajectories converge asymptotically to $x = 0$ without ever leaving the ellipsoid $\mathcal{E}(P, 1)$. □

As the polytope $\mathcal{X}$ has n_x vertices and $\mathcal{D}$ is assumed to have 2^l vertices, the polytope $\mathcal{X} \times \mathcal{D}$ has $n_x \cdot 2^l$ vertices to be tested.

Remark 2. Since Π_d and C_d do not depend on u, quadratic decomposition $\pi_d^\top Q_d \pi_d$ results in a polytopic LMI condition on (x, δ) that holds for all u, i.e. $\mathcal{U} = \mathbb{R}^m$.

Next, a linear SDP program can be formulated for designing a stabilizing gain K. From Madeira (2018), $\Delta \leq 0$ if

$$M_d = \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \geq 0, \tag{42}$$

as this is equivalent to

$$\Delta = SR^{-1}S^\top - Q \leq 0, \tag{43}$$

with $R > 0$. On the other hand, the sufficient condition for stabilization is $\Delta \geq 0$ and it can be verified if we define a new M_d as

$$M_d = \begin{bmatrix} Q + \alpha I & S \\ S^\top & R \end{bmatrix} \geq 0, \tag{44}$$

where $\alpha > 0$ is a real coefficient. By minimizing the function $tr(M_d)$ we might approach some $\Delta \geq 0$, as $tr(M_d) = 0 \Leftrightarrow M_d = 0$ (Yang (1995)) and (44) leads to

$$\Delta = SR^{-1}S^\top - Q \leq \alpha I. \tag{45}$$

Then a systematic procedure for controller design and domain of attraction estimation can be proposed.

SOF Design Algorithm

(1) Consider a nonlinear plant described by (1)-(2) and quadratic functions V and T such as in (26) and (28).
(2) After substitution of (V, T) into (25), determine a decomposition (30)-(31) subject to (8)-(9), and a linear annihilator C_d according to (32).
(3) Initialize a polytope $\mathcal{X} \times \mathcal{D}$ around the origin and determine the n_{xe} vectors a_k that provide a set of hyperplanes.
(4) Specify some $\alpha > 0$ for M_d in (44) and solve the following linear SDP program.

$$\begin{aligned} &\textit{minimize} \ \ tr(M_d), && (46)\\ &\textit{subject to} \ \ (33) \text{ at all vertices of } \mathcal{X} \times \mathcal{D}, && (47)\\ &\qquad\qquad \text{and } (34), (36), (44). && (48) \end{aligned}$$

If feasible, then (Q, S, R, P, N, L_p) is a solution to the SDP program.
(5) Larger domains $\mathcal{X} \times \mathcal{D}$ can be obtained by returning to Step 3 and setting larger values for the vertices of the polytope until Step 4 is no longer feasible.

By applying this algorithm, we guarantee the strict QSR-dissipativity of the plant in $(x, \delta) \in \mathcal{X} \times \mathcal{D}$ and, at the same time, try to ensure asymptotic stabilizability by fulfilling $\Delta \geq 0$.

A polytopic LMI condition for estimating $\mathcal{X}$ for a fixed Lyapunov function was presented in Madeira and Adamy (2016), for instance, where the notion of passivity indices was employed. In that publication, the authors compared the SOS approach with polytopic LMI estimates and verified that the latter, as expected, provide less conservative results than the former. As mentioned before, robust controller design had not been addressed neither in Madeira and Adamy (2016) nor in Madeira (2018). The determination of an ellipsoid $\mathcal{E}(P, 1)$ as an estimate of the domain of attraction was not mentioned in those references neither.

Finally, in Polcz et al (2015) open-loop stability of polynomial and rational nonlinear systems was investigated using an algorithm that automatically generates a vector basis such as π_d in (30)-(31) and matrix functions which play a similar role as Q_d and C_d in (33). Here, we do not implement such an automatic procedure, as it has yet to be adapted to our dissipativity-based framework. Nevertheless, this is certainly our next step in the process of making this new stabilization strategy more suitable for practical applications in the future.

4. NUMERICAL EXAMPLE

The whole controller design procedure can be implemented in MATLAB® with well-known SDP tools (Löfberg (2004), Sturm (1999)). We consider the following version of a nonlinear system which was also analyzed in Baldi (2016)

$$\dot{x_1} = -x_1 + (1 + \delta_1)x_1 x_2 + x_2 u, \tag{49}$$

$$\dot{x_2} = x_1 + 2x_2 + (1 + \delta_2)x_1^2 + x_1^2 x_2 + u, \tag{50}$$

$$y = x_2. \tag{51}$$

Fig. 1 shows the open-loop trajectories of this system for some initial conditions about the origin, with $\delta_1 = -0.05$ and $\delta_2 = 0.05$. Note that the uncontrolled system is unstable.

By applying Theorem 1, we intend to stabilize this system around the origin. Firstly, suppose that $V = x^\top P x$ and $T = x^\top N x$ are quadratic functions such as

$$V(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} v_1 & v_2 \\ v_2 & v_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{52}$$

$$T(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} n_1 & n_2 \\ n_2 & n_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{53}$$

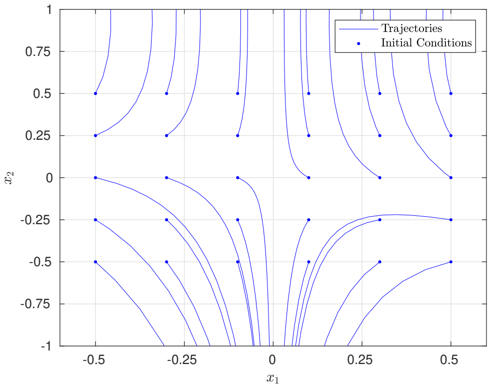

Figure 1. Open-loop trajectories of system (49)-(51).

where (v_1, v_2, v_3), (n_1, n_2, n_3) and (Q, S, R) are real parameters to be determined. We obtain the following function whose nonnegativity in a polytopic domain $\mathcal{X} \times \mathcal{D}$ has to be investigated, whereas $x_d = [x_1\ x_2\ u]^T$ and $\delta = [\delta_1\ \delta_2]^T$,

$$\begin{aligned} t(x_d, \delta) &= t_1 x_1^2 + t_2 x_1 x_2 + t_3 x_2^2 + t_4 x_2 u \\ &+ t_5 x_1 u + R u^2 + t_6 x_1^2 x_2 + t_5 x_1 x_2^2 + t_5 x_1^3 \\ &+ t_5 x_1^3 x_2 + t_7 x_1^2 x_2^2 + t_8 x_1 x_2 u + t_5 x_2^2 u \\ &+ t_8 \delta_1 x_1^2 x_2 + t_5 \delta_1 x_1 x_2^2 + t_5 \delta_2 x_1^3 + t_7 \delta_2 x_1^2 x_2, \end{aligned} \tag{54}$$

with

$$\begin{aligned} & t_1 = -2v_2 - n_1 + 2v_1, t_2 = 2v_2 - 2n_2 - 2v_3 - 4v_2, \\ & t_3 = -n_3 - 4v_3 + Q, t_4 = 2S - 2v_3, t_5 = -2v_2, \\ & t_6 = -2v_1 - 2v_3, t_7 = -2v_3, t_8 = -2v_1. \end{aligned}$$

By setting $\Pi_d x_d + \Pi_2 \pi = 0$, we can determine a vector basis given by

$$\pi = \begin{bmatrix} x_1^2 & x_1 x_2 & x_2 u \end{bmatrix}^\top, \tag{55}$$

$$\pi_d^\top = \begin{bmatrix} x_d^\top & \pi^\top \end{bmatrix}, \tag{56}$$

and a combination (Π_d, Π_2)

$$\Pi_d = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_2 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where $det(\Pi_2) \neq 0$, $\forall (x, \delta) \in \mathcal{X} \times \mathcal{D}$. Then a matrix Q_d is given by

$$Q_d = \begin{bmatrix} t_1 & \frac{t_2}{2} & \frac{t_5}{2} & \frac{t_5(1+\delta_2)}{2} & \Phi & \frac{t_8}{2} \\ \frac{t_2}{2} & t_3 & \frac{t_4}{2} & \frac{t_6+t_7 x_2}{2} & \frac{t_5(1+\delta_1)}{2} & \frac{t_5}{2} \\ \frac{t_5}{2} & \frac{t_4}{2} & R & 0 & 0 & 0 \\ \frac{t_5(1+\delta_2)}{2} & \frac{t_6+t_7 x_2}{2} & 0 & 0 & 0 & 0 \\ \Phi & \frac{t_5(1+\delta_1)}{2} & 0 & 0 & 0 & 0 \\ \frac{t_8}{2} & \frac{t_5}{2} & 0 & 0 & 0 & 0 \end{bmatrix},$$

with

$$\Phi = \frac{t_5 x_1 + t_8 \delta_1 + t_7 \delta_2}{2}.$$

Linear SDP problem (46)-(48) involving a polytopic LMI condition is independent of the control signal u and was proved feasible at the vertices of the region defined by

$$\begin{aligned} \mathcal{X} \times \mathcal{D} = \{(x, \delta)|\ \ & |x_1| \leq 1.4, |x_2| \leq 0.4, \\ & |\delta_1| \leq 0.05, |\delta_2| \leq 0.05\}. \end{aligned} \tag{57}$$

For this polytope $\mathcal{X}$, we have the following $a_k's$ in (29), $k = 1, \cdots, 4$,

$$a_1 = \begin{bmatrix} 0 \\ 2.5 \end{bmatrix}, a_2 = \begin{bmatrix} -0.7143 \\ 0 \end{bmatrix},$$
$$a_3 = \begin{bmatrix} 0 \\ -2.5 \end{bmatrix}, a_4 = \begin{bmatrix} 0.7143 \\ 0 \end{bmatrix}.$$

By proceeding with the algorithm we can determine matrices (Q, S, R, P, N, L_d) that guarantee the feasibility of the problem and a gain that asymptoticaly stabilizes the closed-loop system in $\mathcal{X} \times \mathcal{D}$. For $\alpha = 0.01$ in (44), the following parameters were obtained

$$Q = 166.8087, \quad S = 6.0580, \quad R = 0.2200,$$

$$P = \begin{bmatrix} v_1 & v_2 \\ v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1.0780 & -0.1982 \\ -0.1982 & 6.2865 \end{bmatrix} > 0,$$

$$N = \begin{bmatrix} n_1 & n_2 \\ n_2 & n_3 \end{bmatrix} = 10^{-4} \begin{bmatrix} 0.0022 & -0.0247 \\ -0.0247 & 0.4958 \end{bmatrix} > 0,$$

and

$$L_d = \begin{bmatrix} -0.2026 & 1.8764 & -0.3266 \\ -3.3853 & -1.8508 & -0.2886 \\ -0.1002 & 0.3287 & -0.1064 \\ -0.2738 & 0.4599 & -0.0494 \\ 0.1980 & -11.0958 & -0.4489 \\ 0.0009 & -0.5174 & -0.9112 \end{bmatrix}.$$

With these values of (Q, S, R) we obtain

$$\Delta = SR^{-1}S^T - Q = 0.0067 > 0,$$

and the origin is asymptotically stabilizable by a linear SOF such as

$$K = -R^{-1}S^T = -27.5364 \Rightarrow u = -27.5364 x_2.$$

Furthermore, the ellipsoid $\mathcal{E}(P, 1)$ from (27) is an estimate of the system's domain of attraction. In Fig. 2 the ellipsoid is given by the dashed curve, and for those simulations we considered $\delta_1 = -0.05$ and $\delta_2 = 0.05$.

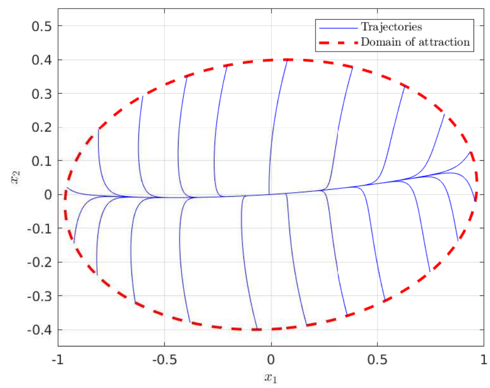

Figure 2. Closed-loop trajectories of system (49)-(51).

5. CONCLUSIONS

This work introduced a new dissipativity-based strategy for local and robust asymptotic stabilization of nonlinear systems by linear static output feedback. In the present paper we have extended the results of Madeira (2018), as the class of the uncertain nonlinear systems was now considered and a simple strategy for estimating the closed-loop domain of attraction was provided. In this article, the Finsler's Lemma and the notion of linear annihilators were applied in order to formulate the problems of robust local dissipativity analysis and linear SOF design as a single linear SDP test containing a polytopic LMI condition. Usually, only the problem of state feedback is treated in such a noniterative and linear SDP fashion, whereas SOF control (a nonconvex problem) is frequently solved through complex iterative algorithms.

We applied the proposed strategy to the stabilization problem of an open-loop unstable system with uncertainties. The simulation results proved the usefulness of the technique. Although the aforementioned system is polynomial, the controller design procedure applies for rational systems as well. Linear static state feedback can also be handled by the same framework, by setting $y = x$.

Future research directions might involve the stabilization problem of nonlinear uncertain systems with input saturation, the case of the LPV systems and anti-windup strategies. A dissipativity-based strategy for dynamic output feedback could also provide an interesting research topic subsequently. Lastly, it could also be considered as another research direction the use of rational Lyapunov functions possibly dependent on the uncertainty and its impact on estimating a domain of attraction and on the computational complexity of the controller design strategy.

REFERENCES

V. Veselý. Static output feedback controller design. *Kybernetika*, 37(2): 205–221, 2001.

C. A. R. Crusius and A. Trofino. Sufficient LMI conditions for output feedback control problems. *IEEE Trans. Autom. Control*, 44(5): 1053–1057, 1999.

P. Apkarian and D. Noll. Nonsmooth H_∞ synthesis. *IEEE Trans. Autom. Control*, 51(1): 71–86, 2006.

P. Gahinet and P. Apkarian. Structured H_∞ synthesis in MATLAB. In *Proc. 18th World Congress IFAC*, Milano, Italy, pg. 1435–1440, 2011.

M. S. Sadabadi and D. Peaucelle. From static output feedback to structured robust static output feedback: A survey. *Annual Reviews in Control*, 42: 11-26, 2016.

J. L. Willems. A system theory approach to unified electrical machine analysis. *Int. J. of Control*, 15(3): 401–418, 1972.

B. Brogliato, R. Lozano, B. Maschke, and O. Egeland. *Dissipative Systems Analysis and Control - Theory and Applications.* Springer-Verlag, London, UK, 2020.

H. K. Khalil. *Nonlinear Systems.* Prentice Hall, Upper Saddle River, NJ, USA, 2002.

D. J. Hill and P. J. Moylan. The stability of nonlinear dissipative systems. *IEEE Trans. Autom. Control*, 21(5): 708-711, 1976.

A. Astolfi, R. Ortega, and R. Sepulchre. Stabilization and disturbance attenuation of nonlinear systems using dissipativity theory. *Eur. J. Control*, 8(5): 408-431, 2002.

Z. Feng, J. Lam, and Z. Shu. Dissipative control for linear systems by static output feedback. *Int. J. Syst. Sci.*, 44(8): 1566-1576, 2013.

R. Ortega and E. García-Canseco. Interconnection and damping assignment passivity-based control: A survey. *Eur. J. Control*, 10(5): 432-450, 2004.

S. L. Shishkin and D. J. Hill. Dissipativity and global stabilizability of nonlinear systems. In *Proc. 34th IEEE Conf. Decis. Control*, New Orleans, LA, USA, pg. 2227-2232, 1995.

D. de S. Madeira. *Contributions to Passivity Theory and Dissipative Control Synthesis.* Ph.D. dissertation, Dept. Elect. Eng., TU Darmstadt, Darmstadt, Germany, 2018.

A. Trofino and T. J. M. Dezuo. LMI stability conditions for uncertain rational nonlinear systems. *Int. J. Robust. Nonlinear Control*, 24(18): 3124-3169, 2014.

D. F. Coutinho, A. Trofino and M. Fu. Guaranteed cost control of uncertain nonlinear systems via polynomial Lyapunov functions. *IEEE Trans. Autom. Control*, 47(9): 1575–1580, 2002.

D. F. Coutinho, M. Fu, A. Trofino and P. Danès. l_2-gain analysis and control of uncertain nonlinear systems with bounded disturbance inputs. *Int. J. Robust. Nonlinear Control*, 18(1): 88–110, 2008.

P. Polcz, G. Szederkényi and T. Péni. An improved method for estimating the domain of attraction of nonlinear systems containing rational functions. *Journal of Physics: Conference Series*, 659(1), 13p., 2015.

D. de S. Madeira and J. Adamy. Output feedback control of rational nonlinear systems: A new approach based on passivity indices. In *Proc. 55th IEEE Conf. Decis. Control*, Las Vegas, NV, USA, pg. 3880–3885, 2016.

S. Azizi. *Robust Stabilization of Uncertain Nonlinear Systems Using Differential Algebraic Representations.* Ph.D. dissertation, Dept. Elect. Eng., Federal University of Minas Gerais (UFMG), Brazil, 2017.

S. Azizi, L. A. B. Torres and R. M. Palhares. Regional robust stabilisation and domain-of-attraction estimation for MIMO uncertain nonlinear systems with input saturation. *Int. J. of Control*, 91(1): 215–229, 2018.

G. Valmorbida and J. Anderson. Region of attraction analysis via invariant sets. In *Proc. IEEE Am. Control Conf.*, Portland, OR, USA, pg. 3591-3596, 2014.

G. Chesi. On the estimation of the domain of attraction for uncertain polynomial systems via LMIs. In *Proc. 43rd IEEE Conf. Decis. Control*, Nassau, Bahamas, pg. 881-886, 2004.

V. M. Haddad and V. Chellaboina. *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach.* Princeton University Press, Princeton, NJ, USA, 2008.

H. R. Pota and P. J. Moylan. Stability of locally dissipative interconnected systems. *IEEE Trans. Autom. Control*, 38(2): 308-312, 1993.

E. R. Rohr, L. F. A. Pereira, and D. F. Coutinho. Robustness analysis of nonlinear systems subject to state feedback linearization. *Revista Controle & Automação*, 20(4): 482–489, 2009.

X.-M. Yang. A generalization of a matrix trace inequality. *Journal of Mathematical Analysis and Applications*, 189(3): 897–900, 1995.

J. Löfberg. YALMIP: A toolbox for modeling and optimization in MATLAB. In *Proc. IEEE Int. Conf. Robotics Autom.*, New Orleans, LA, USA, 2004, pg. 284-289.

J. F. Sturm. Using SeDuMi 1.02, A Matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1–4): 625–653, 1999.

S. Baldi. An iterative sum−of−squares optimization for static output feedback of polynomial systems. In *Proc. 55th IEEE Conf. Decis. Control*, Las Vegas, NV, USA, pg. 3892-3897, 2016.